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EXPLORING ENGINEERING STUDENTS' UNDERSTANDING OF TECHNIQUES OF INTEGRATION

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Abstract

This study explored engineering students' understanding of the techniques of integration in calculus. There were 30 first year engineering students who participated in the project. The concepts were covered as part of a mathematics course at a university of technology in KwaZulu-Natal, South Africa. Activity sheets, constructed with tasks based on integration were administered to the participants. Their written responses, which were used to identify the mental constructions of these concepts, were analysed using the three worlds of mathematics framework and interviews were carried out to clarify the written responses. The discussions and written work indicated that students seemed to be operating in the conceptual-embodied world of cognitive development and students displayed the ability to manipulate symbols and embedded procedures. The findings raised some didactical implications for higher education and also provided applications of the three worlds of mathematics framework.

Keywords: integral calculus, three worlds of mathematics

1. INTRODUCTION

Engineering students learn integration in order to use it as a tool in their respective fields of study. Observations, over years of mathematics teaching, revealed that students struggled to apply integration after learning the theory of integration (Berger 2006, Brijlall & Bansilal 2011, Nguyen 2011). Analysis of the 2015 first semester examinations at a university of technology in South Africa showed that students' performance in electric circuit and heat transfer problems, which require application of integration, averaged 42%. This failure to apply integration reflected poor mental constructions of the concept (Berger, 2006). Nguyen (2011) made a similar observation with regard to the application of integration in physics where students did not understand the meaning of integrands and could not view integration as a summation. Students were, therefore, struggling to use integration as a tool in the engineering field, in spite of having successfully completed the calculus modules.

Furthermore, students' performance in assessments always revealed that students had difficulty in understanding and applying integration. This challenge of poor performance in integration was noticed mainly in the second semester of the first year studies at the South African university of technology at which this study was carried out. The mathematics module that students take in this semester is called 'Mathematics II', the 'II' designating the semester of study. Integration constitutes about 70% of this module, the other topics being hyperbolic functions, partial differentiation and first order differential equations. Students often performed well in the other sections but struggled in integration. The poor performance contributed to a high failure rate in the subject. As a result, many students were blocked progress to advanced levels of study within engineering and, in some cases, they eventually dropped out of the university. We, therefore, became interested in knowing how students developed their knowledge and understanding of an integral and how teaching can be structured in order to enhance students' learning. This study was aimed at answering the primary question: How do students construct mathematical meaning when learning integral calculus? In particular: In what worlds of mathematical thinking do students operate when they internalise integration? How do these worlds influence the learning of the integral calculus? In answering these questions the authors display their scholarship of teaching and research. This paper produces findings which provide empirically based information for improved pedagogy. This applied research study is an original exploration undertaken in order to acquire new knowledge and directed primarily toward lecture-room mathematics practice. It is hoped that the findings from this study, as outlined in the data analysis discussion and conclusions sections of this paper, will inform university lecturing staff with better ideas of instruction. These ideas are empirically based and intended to better the success in student performance. Deeper conceptual grasp of integration will lead to better applications in the workplace. A major desire of many African nations today is to be technologically developed. In South Africa there tends to be an acute shortage of skilled manpower in the field of engineering and technology. The fundamental importance of mathematics to humans could be explained in terms of the interrelationship between mathematics and development of humans to advance the cause of humans. The topic of integration in this study has numerous applications in the manufacturing industry. When engineers model their products they use integration techniques (like volumes of solids of revolution, arc length of shapes, etc.). This study, hence provides new knowledge to both educational theory and practice that keeps the engineering workplace in mind.

2. THEORETICAL ANALYSIS

Tall (2004b) differentiates between the three stages or three worlds of mathematics (TWM) through which mathematical learning develops. These stages are the conceptual-embodied or embodied, the proceptual-symbolic or symbolic and the axiomatic-formal or formal world.

The embodied world refers to that stage of learning where operations are based on human perceptions and actions in a real-world context but it also includes imagining the properties in the mind (Tall 2004b). In this level of cognitive development, the learner's conceptions are dependent on the properties of objects and reflections on those properties (Tall, 2007). At this level a learner will still be expected to provide solutions through imagining a situation occurring and thinking through the consequences. Hence, this level includes enactive and iconic examples with an increasing inclusion of visual and spatial imagery (Tall, 2002). The knowledge of a physical drawing of a straight line, for example, will provide ability to conceptualise a complex fact that a line has length but no breadth (Tall, 2002).

The proceptual-symbolic or symbolic world, grows out of the embodied world and it involves the role of symbols and symbol-processing in different aspects of mathematics (Tall, 2004b). In this world, actions "are encapsulated as concepts by using a symbol that allows us to switch effortlessly from processes to do mathematics to concepts to think about" (Tall, 2004b: 5). It is the world "where actions, processes and their corresponding objects are realised and symbolised" (Stewart & Thomas, 2008: 205). This level develops through several distinct stages. Examples are: arithmetic calculations which lead to algebraic manipulations then to limit concepts. Another example will be in operations, where learners start with normal addition and subtraction, then multiplication and division and other related operations. This reaches its peak when differentiation and integration are included.

The axiomatic-formal or formal world is where thinking is predicated for definitions and proofs (Tall, 2007). It begins with formal set-theoretic definitions which are constructed through deductions made from the embodied experience. These definitions are then formulated into a complete systematic axiom theory. Formal proofs are subsequently used to construct meaning from set-theoretic definitions and other properties deduced, using formal proofs (Tall, 2002). In this case, the (non) existence of a derivative, for example, is established through proof. At this level mathematical conception is based on logical reasoning (Tall, 2008).

Tall (2008) purports that internalising an action into a process and encapsulating it into an object, with connections to other knowledge within a schema, is a form of compression. Compression is when the brain synthesises pieces of information "by connecting ideas together into thinkable concepts" (Tall, 2008:10). He further argues that there is a correspondence between the symbolic and the embodiment compression. Both types of knowledge development start with procedures and for each subsequent stage in the symbolic compression, there is an embodied precept. To highlight how this applies in our study we provide the case of integration by parts. In this case, knowledge development would commence with the procedure of decomposing the integrand into two appropriate parts. Thereafter, for each subsequent stage in the symbolic compression (for both differentiation and integration) there is an embodied precept. These precepts arise out of previous notions of differentiation and integration. Another South African study (Mholo & Schafer, 2013) used this terminology of precept but referred to it as preconception arising out of the work of Mc Gowen & Tall (2010).

3. RESEARCH METHODOLOGY

This study aimed to explore the meaning of integration from students' perspectives, within the TWM and probably make discoveries that could contribute to the development of empirical knowledge about conceptual development of integration, for such a group of students. This was a single qualitative case study research to investigate concept development of integral calculus, for first-year engineering students, at a South African university of technology. The participants were subjected to activity sheets with integration tasks. The written responses to these tasks were analysed and we present the analysis in terms of the mathematical concepts involved. The technique used to collect data was the focus group discussions where 30 students worked collaboratively, in groups of six, to solve given problems. Focus groups are defined as "in-depth interviews employing relatively homogeneous groups to provide information around topics specified by the researchers" (Smithson, 2008:358). The lecturer served as a "soft scaffold", as defined by McCosker & Diezmann (2009), through asking probing questions and providing explanations whenever necessary. Focus groups provided data that was mainly from the students' voice.

Two hour discussions were held on two Friday afternoons and were video-recorded. Photographs of students and their work were also taken. In order for all participants to be active, the size was kept to four members per group. According to Smithson (2008), smaller groups yield relevant data and allow space for all participants to express themselves. The analysis of discourses from these groups will be discussed.

3.1. Ethical issues

Participation was completely voluntary and students could withdraw any time when they so wished. This explanation was captured in a letter of consent that was read to them and which they all signed. The letter also contained a brief explanation and context of the project. It notified the participants of the methods through which data would be collected and assured them that their identities would be protected.

3.2. Discussion of data

The discussion of data is divided according to the development of mathematical concepts in accordance with the TWM, embodied, symbolic and formal.

3.2.1. Conception of inverse of the chain rule

With regard to reversing the chain rule, students were requested to work out (a) $\int \frac{\ln x}{x} dx$ and (b) $\int \frac{1}{x \ln x} dx$ in each case to justify the approach chosen. Both items (a) and (b) required students to identify functions multiplied in the integrands. In (a), for example, the integrand consisted of the product of $\frac{1}{x}$ and $\ln x$. Although this task could be solved using the u-substitution technique, all groups opted to use the table of standard integrals. Next, is an extract from a discussion of the solution to (a) within one of the groups, call it Group 1.

L1: Maggie: *This thing* (pointing at the integrand) *is, you see,* $\frac{1}{x}$ *times* $\ln x$
L2: Roy: *So we agree that we are doing this rule?*
(At this stage Roy pointed at the first standard integral in the data sheet of their Study Guide. This standard integral is
$\int [f(x)]^n f'(x)\, dx = \frac{1}{n+1}[f(x)]^{n+1} + C, n \neq -1.$)
L3: Roy: *We are going to say, the answer is equal to 1 over…ehh what is "n"?…it is* 1*, so it is* 1 *plus* 1,
L4: *times,…what is f(x)?…it is ln x,* 1 *plus* 1, *plus C.*
L5: *So the final answer is 1 over 2 ln x squared plus C.*
Extract 1: Group 1's conversation about Item 4(a)

Maggie was the first to comment on the way forward in solving this problem. She correctly identified the two functions multiplied within the integrand as $\frac{1}{x}$ and $\ln x$ (line 3 of Extract 1). Roy then took the lead in discussing the solution further, identifying the standard integral applicable. We noted that neither Maggie nor Roy explicitly categorised the functions $\frac{1}{x}$ and $\ln x$ to $f(x)$ and $f'(x)$ in line with the standard integral chosen. Roy solicited the group's endorsement by inserting leading questions such as: "what is 'n'?" and "what is 'f(x)'?" within his presentation (lines 3 and 4 in Extract 1). The whole group joined him in answering these "sub-questions". As such, although his voice was dominant, answers were provided in chorus form. This group ultimately presented a consensus solution as follows:

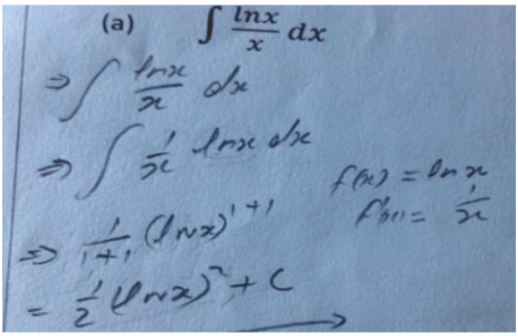

Figure 1: Group 1's response to item 4(a)

The presentation by this group indicated that they had conceptually embodied the action of reversing the chain rule into a process. According to Tall (2007), conceptual-embodiment is when an individual's mental constructions are guided by in-depth perceptions and reflections on the nature or structure of a concept and various representations of such a concept. After tackling many tasks on integration, using a variety of techniques, the students in this group immediately identified the technique of integration required for this particular problem. The identification of the technique emanated from the identification of $f(x)$ and $f'(x)$ in the integrand, which was done mentally, as can be inferred from the verbal interactions in Extract 1 above.

Writing the integrand as a product in line 2 of Figure 2 indicated that students perceived the nature of the integrand to be a product of two functions, thus expressing it in the exact form of a standard integral. The application of the identified standard integral further required conceptualisation of a composition within the product. In this case, it appeared that Roy figured out that lnx was the composite function with an exponent equal to 1, as stated in line 4 of both oral and written extracts (Extract 1 and Figure 2). It was also important to work out the derivative of $f(x) = lnx$, the inside function in the composition, checking whether the format in the standard integral was satisfied. The omission of a constant of integration in line 3 of Figure 2 was considered insignificant since they included it at the end.

Of further significance was the representation of the final answer, where lnx was put within brackets. Such representation indicated understanding of how a concept is to be represented. This illustrates that the group members were operating in a proceptual-symbolic world of mathematics. A student without that level of understanding may fail to present a correct solution to a problem of this nature. Suzan, for example, was one student in the group, who successfully conceptualised the composition within the integrand but

provided $\frac{lnx^2}{2}$ +C as her final answer. Although she understood that "the $f(x)$ was lnx, and 'n' was 1" and also knew the procedure for integrating $[f(x)]^n$, she had not conceptualised the fundamental difference between In x^2 and In^2x. According to the standard integral the group was using, they were supposed to square lnx, that is $(lnx)^2$. Written without brackets then $(lnx)^2 = lnx$ x $lnx = ln^2x$. On the other hand, $ln\ x^2$ is actually equal to In $(x$ x $x)$, which is not what the standard integral dictates. Maggie indicated the error to Suzan who then changed her answer and made it look like Roy's.

3.2.2. The 'u-substitution method'

The same functions, $ln\ x$ and x, were used in item 4(b) but were combined differently. Item 4(b) was $\int \frac{1}{xlnx} dx$. Group 2 took time reflecting on this item, exploring various approaches to use, until Sello identified the 'u-substitution method' as appropriate. It seemed that the rest of the group were not familiar with this approach, although it was one of the techniques that had been discussed during lessons in class. This is evident from the following discussion:

Line 1: **Thabo:** *What are we going to do? Maybe use integration by parts.*
Line 2: **Sello:** *Wait, Let us use substitution.*

Line 3: **Pete:** *Which one?*
Line 4: **Sello:** *Where you use ….* (the rest of the group says: use 'u' and 'v') *...no, that is integration by parts. Substitution is where you convert*
Line 5: **Pete:** *You use that in differential equations*
Line 6: **Sello:** *No no no, this is integration by substitution, you don't know it?*
Line 7: **Pete:** *Write it down*
Line 8: **Sello:** *It is not differential equations. Wait, wait, wait…*
Line 9: **Thabo:** *Show us the formula that you use.*
Line 10: **Sello:** *You substitute…..wait…it is almost like integration by parts, but it is not it exactly. You put 'u' equal to something, but I cannot remember well. Let us see,…*

Extract 2: Group 2's conversation about Item 4(b)

The members of this group were imagining the properties in their minds and expressed such imaginations verbally. This hinted that these students were working in an embodied world of mathematics. Sello displayed some degree of confidence in his chosen approach. He was clear in his mind that the 'u-substitution method' differed from integration by parts. He vehemently rejected the group's suggestion to use 'u' and 'v', as indicated in line 4 of Extract 3. Sello eventually recalled how to proceed with the 'u-substitution method' in this item. He started by splitting the integrand into a product, that is, $\int \frac{1}{x}\cdot\frac{1}{lnx}dx$. He then let $u = ln\,x$. Differentiating, he obtained $\frac{du}{dx} = \frac{1}{x}$. He proceeded to make dx the subject of the formula, obtaining $dx = xdu$. The next step was substituting for $ln\ x$ and dx into the integral. Sello wrote $\int \frac{1}{xu}.xdu$ and after simplifying, the integral reduced to $\int \frac{1}{u}du$. This was a simpler integral to work out, giving $ln\,u + C$ as the answer. The last step was to substitute the u in this integral, yielding $ln\,(ln\,x) + C$ as the final answer. Of course Sello did not state the condition that $ln\,x > 0$.

The 'u-substitution method' is used to transform an integral to another integral that is easier to work out. It is theoretically based on the chain rule for differentiation which states that $^d/_{dx}\ [f(g(x))]=f'(g(x)).g'(x)$. Integrating this equation yielded $\int f'(g(x)).g'(x)dx= \int {}^d/_{dx}[f(g(x))]dx=f(g(x))$. Letting $u=g(x)$, thus $^{du}/_{dx}=g'(x)$, transforms the integral to $\int f'(u)\ {}^{du}/_{dx}\ dx=f(u)$. If we re-write $du/dx=g'(x)$ as $du=g'(x)dx$, the integral becomes $\int f'(u)\ du=f(u)$, which is a simpler integral in the variable 'u'.

Thabo asked for a formula that Sello was using (line 9 of Extract 3). Pete requested Sello to write down the substitution to which he was referring. These two students could only carry out the required integration by reacting to explicit external cues outlining steps to follow.

Sello gave an explanation for all the steps he was writing. Unlike on paper, where he had first split the product in the integrand, on the board he started by doing the substitution directly. This move, however, did not create any confusion since most of the students had already attempted this item. Sello's presentations, both on paper and on the board, indicated the presence of reflections and perceptions on the properties of the integral concerned. He demonstrated greater power and precision when manipulating symbols. This seemed to suggest that Sello was in the proceptual-symbolic or symbolic world of mathematical conception.

What we observed for this task was that Sello took the lead and drove the discussion to present the solution. This is one of the drawbacks of group work (Brijlall 2014). However, the flip of the coin is that the others in the group could be peer taught into the correct path. Otherwise, it could have been a long time before they might have arrived at any correct outcome.

3.3.3. The multiplicative inverse of a function

It was interesting to note that Group 3 used a different approach to solve item 4(b). Zola, who was leading discussions for this item, was strongly challenged by her group peers when she presented the solution. Zola started by claiming that the standard integral applicable in this item was $\int f'(x)/f(x)\, dx = ln|f(x)| + C$. She proceeded to separate $1/x$ and $ln\, x$ within the integrand and said:

Zola: *Here is the rule, it says f'(x) over f(x) is the answer. Isn't when we split here it's going to be 1 over x times 1 over lnx. When we differentiate what will the derivative of lnx be?*

The group then asked her to identify the $f(x)$ in the problem. When she pointed at the lnx in $^1/_{lnx}$, the other students disputed that claim. An interesting dialogue ensued, with Zola attempting to defend her position:

Line 1: **Tebogo:** *It should be the whole thing as a function (referring to $^1/_{lnx}$).*
Line 2: **Mike:** *It will be 1 over 1, and then 'x' will go above the line.*
Line 3: **Zola:** *We are using this rule which says f'(x) over f(x). Then here, f(x)…they say 1 over. So our f(x) will be taken as….our f(x) is ln x.*
Line 4: **Daniel:** *1 over ln x will not yield 1 over x, it will be x because it will be 1 over 1 over x. Then x will go above the line (Tebogo and Mike agreed with him).*
Line 5: **Mike:** *There are two things here. Our f(x) should be 1 over ln x.*
Line 6: **Tebogo:** *Here is the rule, bafowethu* (brothers), *this first one.* (Here, Tebogo pointed at $\int [f(x)]^n f'(x)\ dx = 1/(n+1)\ [f(x)]^{(n+1)} + C\ ,\ n\ -1$ in the tables of standard integrals).
Line 7: **Mike:** *It is not the first rule. I know the answer. It is not on the first rule.*

Extract 3: Group 4's conversation about Item 4(b)

Two misconceptions were displayed. Firstly, the three students realised and agreed that ${}^{d}/_{dx}\ (lnx)=1/x$ but were struggling to conceptualise 1/lnx as a composite function $f[g(x)]$ where $f(x)={}^{1}/_{x}$ and $g(x)=lnx$. They viewed $1/lnx$ as a single entity (see Lines 1 and 5 of Extract 4) and as such they could not detect the reversal of a chain rule in this item. With that fixation, they proceeded to differentiate ${}^{1}/_{lnx}$ where the second misconception was displayed. Although the derivative of the reciprocal $1/lnx$ was not required for this item, it was noted that students showed gaps in their knowledge of differentiation. The three students agreed that ${}^{d}/_{dx}\ ({}^{1}/_{lnx})={}^{1}/_{(1/x)}=x$, evidence of an error in differentiating a multiplicative inverse of a function.

This showed that the prerequisite knowledge necessary for integration was lacking. Firstly, they could have exploited the quotient rule to arrive at the legitimate outcome or secondly, they could have used the chain rule. As indicated in Sello's presentation above, taking the u-substitution path, the following would have been the solution:

$$
\begin{aligned}
&Let\ u=lnx\\
&{}^{du}/_{dx}={}^{1}/_{x}\\
&xdu=dx\\
\int {}^{1}/_{xlnx}\,dx &= \int {}^{1}/_{xu}\times xdu\\
&=\int {}^{1}/_{u}\,du\\
&=ln|u|+C\\
&=ln|lnx|+C;x>1
\end{aligned}
$$

The $y=lnx$ function and the area bounded by it and the x-axis appears in practical problems in engineering, especially in the design and modelling of real world problems. The students' reflections on their imagination of the integration processes, displayed their existence in the embodied world of mathematics.

3.3.4. Conception of integration by parts

To explore students' conceptual understanding of the use of integration by parts, they were given carefully selected tasks. One of the tasks required students to evaluate $\int ln(x^2-x+2)dx$. They were not told which technique to use.

This problem provided a case where the integrand did not feature in the table of standard integrals. At this level of study students had dealt with the derivative of lnx and knew that it was ${}^{1}/_{x}$. Other functions in the same category as this one are inverse trigonometric functions like $sin^{(-1)}x$, where ${}^{d}/_{dx}(sin^{(-1)}x)$ is known to be $1/\sqrt{(1-x^2)}$, while the integral of $sin^{(-1)}x$ is not readily known. Such a problem requires the use of integration by parts to solve. Xola, who was working with Lwazi, readily identified the technique applicable to this problem. He could not explain much about his choice and instead chose to lead his partner through the solution. The following is the conversation they had:

Line 24: **Xola:** *This is gonna be integration by parts. We say "u" will be* $ln(x^2-x+2)$ *and "dv" will be "dx". Do it. Use* ln, x^2
Line 25: **Xola:** *Ya, write this as "u" and "dv" will be "dx". We will get at the end but let's give it a try.*
Line 26: **Lwazi:** *Hey, I am not sure about this!*
(At this stage Lwazi proceeded to differentiate $ln(x^2-x+2)$)

Line 27: **Xola:** *No no no, It will be 2x , du will be 2x-1...No no my friend, if you are integrating this you write….,*
Line 28: **Lwazi:** *Differentiation, we are not integrating. If you differentiate this, what is the answer?*
Line 29: **Xola:** *Yes differentiating I agree. Let me write it. It will be* $2x-1$ *over* x^2-x+2
Line 30: **Lwazi:** *If I am saying this,1 over* x^2-x+2, *times 2x-1, am I wrong if I say so......?*
Line 31: **Xola:** *Well, it is the same, now continue. Write, dv =dx and therefore v=x because there is a 1 here and the integral of 1 is "x". Then go to the formula:*
Line 32: **Lwazi:** *I am not sure about this bra..*

Extract 4: Conversation between Lwazi and Xola

Lwazi seemed to know that when using integration by parts, the "u" should be differentiated in order to determine the "du". With reference to his confession in line 26, Lwazi's answer was a mere response to the procedure of integrating by parts that he knew. He, nonetheless, displayed conceptual-embodiment of the chain rule for differentiation. He defended his approach when Xola stopped him as he was writing out the derivative of $ln(x^2-x+2)$.

Although Xola displayed efficiency in choosing the suitable procedure to use for this task, an example of compression of aspects into thinkable concepts according to Tall (2007), the above extract reveals some gaps in his foundational conceptions. Firstly, he was using the terms integration and differentiation interchangeably, which is mathematically inaccurate. Lwazi corrected that error in line 28 when he emphasised that they were differentiating (the 'u') and not integrating. Secondly, he wanted to insist on a single representation of the derivative of $ln(x^2-x+2)$. He did not wait for Lwazi to finish writing but assumed that it would be incorrect and so offered his "correct version" of the derivative. Lwazi then asked whether the derivative could not be equally written as a product of $1/(x^2-x+2)$ and $2x-1$ (Line 30)? Xola continued to guide Lwazi in the use of integration by parts but struggled to manipulate the subsequent integral that arose (see Item 5.5 in Figure 2).

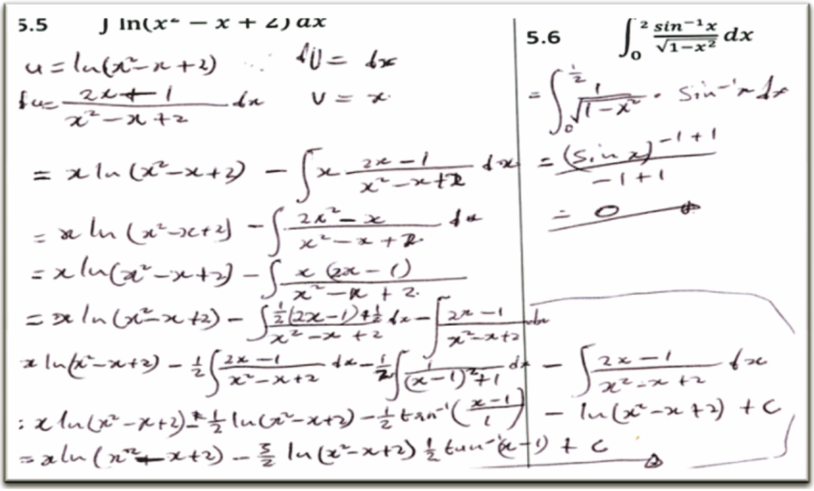

Figure 2: Lwazi and Xola's solution on integration by parts

Xola and Lwazi applied the rule of integration by parts correctly. The first four lines of their solution indicated a proceptual-symbolism, which according to Tall (2007), is the use of symbols as thinkable concepts. Tall (2007) refers to an elementary procept as being the "combination of symbol, process, and concept constructed from the process" (p.2). In this instance, the students moved flexibly between differentiating the 'u' and integrating 'dv' and structured their results correctly, in line with the rule for integrating by parts. They, therefore, possessed this elementary procept which enabled accuracy in working out the components of the integral.

According to Gray and Tall (1994), individuals possess a precept if they have mastered the collection of elementary procepts with the same output concept. In this case, that would refer to mastery of all embedded integration techniques to solve a sum. Regarding Xola and Lizwi, they struggled to evaluate $\int \frac{2x^2-x}{x^2-x+2}dx$ that arose when integrating by parts. They could not recognise equal degrees for the numerator and denominator, thus a need to first simplify by dividing the two expressions. As a result, their final solution was incorrect. We observed for item 5.6 (see Figure 2), that despite recalling the elementary precept of the integral of a polynomial term and applying the power rule, it does not necessarily lead to a correct solution. Here the students demonstrated that they knew that $\int x^n dx = \frac{x^{n+1}}{n+1}$ but applied it incorrectly to the inverse sine function. This is probably due to the unpopular use of the notation for the inverse sine function. This demonstrates that lecturers should adopt the use of arcsin for the inverse trigonometric functions rather than the one used in Figure 2.

Xola and Lwazi had skipped item 5.4 which was $\int tan^{(-1)}(3x)dx$ but after working on item 5.5, they realised that the two problems required the same technique. What was noted was that Lwazi was more forthcoming and he voluntarily did all the writing. We only present their solution in Figure 3, since all their discussions were the steps that they eventually wrote down.

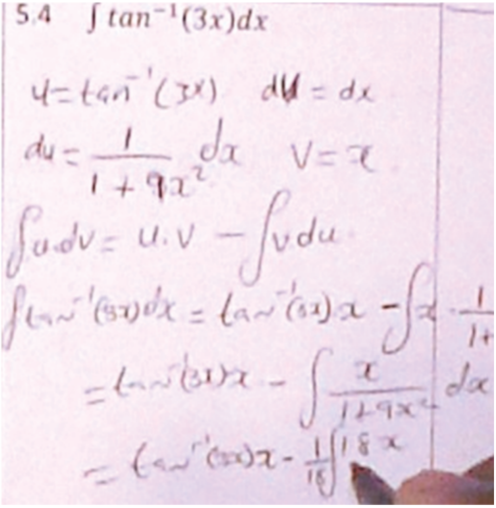

Figure 3: Whiteboard work emanating from discussions between Xola and Lwazi

We note that the rule of integration by parts was applied correctly, that is, a correct separation of the integrand into $u=tan^{(-1)}\,3x$ and $dv=dx$. Nevertheless, both students could not realise the mistake when determining ${}^{d}/_{dx}\ tan^{(-1)}(3x)$. They wrote $1/(1+9x^2)$ instead of $3/(1+9x^2)$ in line 2 of Figure 3 above. This oversight persisted even when we tried to draw it to their attention, but they eventually recognised their mistake as can be derived from the conversation below (Extract 5):

Researcher: *There is a 3 here, what did you do with it?*
Lwazi : *We know that* ${}^{d}/^{dx}\ (tan^{(-1)}(x))=1/(1+x^2)$*, so here it is* $tan^{(-1)}(3x)$ *so it is* $1/(1+(3x)^2)$.
Researcher: *What if it was* $tan^{(-1)}(x^2)$*?*
Xola: *It will be 1 over , in the place of 'x' we put* x^2 *, so it will be 'x' to the power 4.*
Researcher: *Is that all?*
Xola: *Ya…oh, there is an error here it is supposed to be times* $2x$ *. Oh, so we are supposed to say times 3.*

Extract 5: Conversation between Lwazi and Xola

Xola and Lwazi responded interchangeably, an indication that both of them were equally confident of the approach they were using. Their presentation indicated that they had embodied the procedure of integration by parts. According to Jojo, Maharaj & Brijlall (2013), students operating in the action stage, view a mathematical procedure as a series of individual steps. They focused mainly on producing a correct solution with less justification on how they produce such a solution. In addition to focusing on the steps, Xola and Lwazi displayed gaps in some underlying procedures required for this technique. In the second line of Figure 3, for example, having correctly set $tan^{(-1)}(3x)$ as a "u", Lwazi could not recognise the need to apply the chain rule for differentiation. Xola only realised the error when probed and given $tan^{(-1)}(x^2)$ as scaffolding.

4. CONCLUSIONS AND RECOMMENDATIONS

Most of the discussions and written work indicated that students seemed to be operating in the conceptual-embodied world of cognitive development. This was found when Zola presented and argued her case for the composition of functions when integrating and adopting a reversal of the chain rule. Even Lwazi and Xola indicated a proceptual-symbolism when processing integration by parts. For the given integrals, some students could reflect and perceive the integrand properties and thus could decide on the correct technique to employ. The majority of presentations also revealed that most students struggled to interpret compositions, particularly in a case of an inverse function. Knowledge gaps in differentiation, symbolic notation and integration were also identified as having an effect on students' success to solve integrals.

When presented with integrals that required the reversal of the chain rule, students displayed mental constructions that were based on detailed discernments and considerations of the functions involved. Students could mentally identify the $f(x)$ and $f'(x)$ in the integrals $\int {}^{lnx}/_{x}\, dx$ and $\int {}^{1}/_{(x\, lnx)}\, dx$, hence decided on an appropriate technique to use. Results for $\int {}^{lnx}/_{x}\, dx$ indicated the recognition of the exponent '1' in $f(x)=lnx$, thus correctly providing ${}^{1}/_{2}\, (lnx)^2+C$ as an answer. Misconceptions with symbol syntaxes resulted in some students presenting lnx^2 instead of $ln^2\, x$, an indication of weak precepts of algebraic symbols. When working in focus groups, basic errors such as the omission of constants of integration were not displayed.

The results indicated that students employed two approaches when dealing with the integral $\int {}^{1}/_{(x\, lnx)}\, dx$. The first approach was the use of the 'u' substitution method, while other students viewed the given integral as an integral of a multiplicative inverse for $f(x)=lnx$. The ability to transform integrals from the 'x' to the 'u' variable indicated proficiency with symbol manipulation. The 'u-substitution' requires accurate analysis of a composition in the integrand and correct performance of differentiation. While signals of gaps were noted in handling restrictions of the domain of the function $y=lnx$, most observed responses indicated that students were reflecting on the properties of the integrands and could also handle symbolical representations. Students were, therefore, deemed to be operating in both conceptual-embodied and proceptual-symbolic worlds of mathematical meaning.

Alternatively, the results indicated challenges for some students who opted for the approach of viewing $\int {}^{1}/_{(x\ lnx)}$ dx as $\int (f'(x))/(f(x))\ dx$. The failure to conceive the composition in ${}^{1}/_{lnx}$, which is $(lnx)^{(-1)}$, as well as errors in determining ${}^{d}/_{dx}\ ({}^{1}/_{lnx})$ indicated weak conceptual-embodiment. Firstly, students could not perceive the embedded representation of a power in ${}^{1}/_{lnx}$. As a result, they were persistent in differentiating ${}^{1}/_{lnx,}$ instead of lnx . Such an error signified poor perceptions of properties of integration. Secondly, gaps were also displayed in the underlying concepts of differentiation as students were insisting that ${}^{d}/_{dx}({}^{1}/_{lnx})=1/(1/x)=x$, indicating a lack in the prerequisite knowledge necessary to carry out integration.

Nonetheless, results showed that the levels of operation for students were varied. The presentation and argument by Zola indicated advanced entrenching, in both the embodied and symbolic worlds of thinking. When Zola could not justify her approach verbally, she opted for symbolic representation. Her expression of the integral $\int {}^{1}/_{(x\ \ lnx)}\ dx$ as $({}^{1}/_{x}\div lnx)dx={}^{1}/_{x}\times{}^{1}/_{lnx})dx={}^{1}/_{x}\times[lnx]^{(-1)})dx$ indicated an in-depth understanding of the integral. In addition, she succeeded to use the language of mathematical symbols to convey her thoughts. She was using symbols as thinkable concepts. A similar observation was made with respect to Thembi when working with ${}^{ex}/_{(ex+1)}\ dx$. Thembi could not state the relationship between the numerator and denominator functions verbally but relied on symbols to explain her line of argument.

With regard to integration by parts, students displayed the ability to manipulate symbols and embedded procedures. For example, the technique of integration by parts gives rise to a 'u' and a 'dv' which require differing operations. Students managed that section of the task successfully. The tendency was to focus on step-by-step procedure to get a solution, subsequently omitting critical underlying aspects such as proper notation and correct differentiation. Students could work with symbols, the actual procedure and emerging concepts within the technique of integration by parts. They were using symbols as thinkable concepts, thus operating at a proceptual-symbolic world of mathematics learning (Tall 2007). In short, students possessed the elementary procept for the technique of integration by parts.

Challenges observed included: (1) misconceptions with the syntax of symbols where some students expressed $ln^2\ x$ as lnx^2 ; (2) failure to recognise embedded compositions such as $(lnx)^{1}$ and $(lnx)^{(-1)}$ when re-writing the integrals as $\int (lnx \cdot {}^{1}/_{x})dx$ and $\int ({}^{1}/_{lnx} \cdot {}^{1}/_{x})dx$ respectively; (3) errors in using the symbols of integration and basic differentiation rules and (4) gaps in the underlying knowledge and skills, such as the use of the chain rule in differentiation. All these findings should be kept in mind when teaching this section of integration. Hence, the findings of this study have serious implications for pedagogy which also helped contribute to the general theory of the TWM.

Furthermore, integration is a topic in Calculus that has numerous applications in engineering. Students at university require a sound conceptual understanding of when using the techniques of integration. In scientific investigations, physical interpretations are often attached to areas.

In order to determine areas of irregular shapes the techniques on integration are adopted. One example of this occurs in the theory of elasticity (Swokowski 1984). In order to test the strength of a material an engineer records values of strain that corresponds to various loads (stresses). Two curves are obtained and the area of the region enclosed by these curves can be determined by integration techniques which are dealt with in this paper. Hence the new ideas provided in this study are intended to provide new ideas to improve the understanding of integration techniques so that engineers are able to better facilitate the design of the projects they encounter in the workplace.

For future studies, the following question is recommended for further investigation: How will the prior introduction of graphical functional representation affect conceptual understanding of the integral concept by engineering students?

The contention is that graphical representations of functions will result in students operating in the object stage of integration. Students will be able to link an integral to the area concept, as well as incorporate the underlying restrictions when dealing with functions such as $f(x)=lnx$. Again, the suggestion is that a quasi-empirical research method be adopted using control and experimental groups.

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