Stochastic mean absolute deviation model with random transaction costs: securities from the Johannesburg stock market

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Abstract: We propose a multi-stage stochastic mean absolute deviation model with random transaction costs in optimal portfolio selection. We take implicit costs incurred in trading as our transaction costs. The multi-stage stochastic model captures risk due to uncertainty, as well as implicit transaction costs incurred by an investor during initial trading and subsequent rebalancing of the portfolio. We apply the model to securities on the Johannesburg stock market and find out that implicit transaction costs are at least 14.3% of returns on investment.

Keywords: stochastic mean absolute deviation; random transaction costs; uncertainty; stochastic programming; portfolio rebalancing.

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1 Introduction

Methods of operations research, especially mathematical programming methods, are receiving broader acceptance in the economic and financial industry. The increasing complexities and inherent uncertainties in financial markets have led to the need for mathematical models supporting decision-making process. This research intends to address the portfolio selection problem by applying stochastic programming. We construct a multi-stage stochastic mean absolute deviation (MAD) model that captures asset returns and risk due to uncertainty.

The MAD model was first proposed by Konno and Yamazaki (1991), in deterministic form, as an alternative to the famous and widely used mean-variance model by Markowitz (1952). We have adopted stochastic programming as it has a number of advantages over other techniques. Firstly, stochastic programming models can accommodate general distributions by means of scenarios. We do not have to explicitly assume a specific stochastic process for the securities' returns, but we can rely on the empirical distribution of these returns. Secondly, they can address practical issues such as transaction costs, turnover constraints, limits on securities and prohibition of short-selling. Regulatory and institutional or market-specific constraints can be accommodated. Thirdly, they can flexibly use different risk measures. This presents the choice to optimise the appropriate risk measure or utility function.

MAD is a dispersion-type risk linear programming (LP) computable measure that may be considered as an approximation of the variance when the absolute values replace the squares. Konno and Yamazaki (1991) propose the MAD model as a risk measure to overcome the weaknesses of the variance. This MAD model is equivalent to the mean-variance model by Markowitz (1952) if the assets' returns are multivariate normally distributed. However, the MAD model is a special case of piecewise linear risk model which is fast in optimising portfolios by means of LP unlike the mean-variance model that requires quadratic programming. The use of a linear model considerably reduces the time needed to reach a solution, thereby making it more appropriate for large-scale portfolio selection. It makes extensive calculations of the covariance matrix unnecessary, as opposed to the mean-variance model. The MAD model is also sensitive to outliers in historical data (Byrne and Lee, 2004).

The MAD model has its short comings. Ignoring the covariance matrix can cause great estimation risk (Simaan, 1997). MAD also penalises not only the negative deviations, but also the positive deviations. Nevertheless, investors prefer higher positive deviations and avoid lower negative deviations in portfolio returns. Fama (1965) explains that making a distinction between positive and negative returns is necessary if portfolio returns are asymmetrically distributed and stock returns are skewed. Fishburn (1977) introduced downside-risk measures to deal with such problems. The advantage of downside-risk measures is that they only penalise returns below a given threshold level specified by the investor. Michalowski and Ogryczak (2001) extend the MAD model to incorporate downside-risk aversion. Hoe et al. (2010) make an empirical comparison of the mean-variance, MAD, minimax and mean-semi-variance models in portfolio optimisation. They compare the portfolio compositions and performance of different optimal portfolios by using data of monthly returns of 54 stocks included in the Kuala Lumpur Composite index from January 2004 to December 2007. They find the most risk portfolio to be that of the mean-variance model while the MAD model generated portfolio is the least risky. The minimax model shows the highest performance, followed by the MAD model, and the mean-variance gives the least performance. However, despite the good performance by the minimax model, it has its disadvantages. Because of its objective to minimise maximum loss, minimax is sensitive to outliers in historical data (Young, 1998).

Xiao and Tian (2012) estimate implicit transaction cost in Shenzhen A-stock market using the daily closing prices, and examine the variation of the cost of Shenzhen A-stock market from 1992 to 2010. They use the Bayesian Gibbs sampling method proposed by Hasbrouck (2009) to analyse implicit costs in the bull and bear markets. Hasbrouck (2009) incorporates the Gibbs estimates into asset pricing specifications over a historical sample and find that effective cost is positively related to stock returns. Kozmik (2012) discusses an asset allocation with transaction costs formulated as a multi-stage stochastic programming model. He considers transaction costs as proportional to the value of the assets sold or bought, but does not consider implicit trading costs in the model. He employs conditional-value-at-risk as a risk measure. Moallemi and Saglam (2011) study dynamic portfolio selection models with Gaussian uncertainty using linear decision models incorporating proportional transaction costs. They assume that trading costs such as bid-ask spread, broker commissions, and exchange fees are proportional to the trade size. However, as stock prices follow a random-walk process, this would result in trading costs fluctuating due to a number of factors that include liquidity of stocks, market impact and so on. Considering proportional transaction costs in an uncertain environment does not provide good estimate of trading costs, especially implicit transaction costs. Brown and Smith (2011) study the problem of dynamic portfolio optimisation in a discrete-time finite-horizon setting, and again, consider proportional trading costs. Lynch and Tan (2006) study portfolio selection problems with multiple risky assets. They develop analytic frameworks for the case with many assets taking proportional transaction costs. Korn (2011) studies continuous-time portfolio optimisation and considers proportional transaction costs. Cai et al. (2013) examine numerical solution of dynamic portfolio optimisation with transaction costs. While transaction costs are broad and includes explicit as well as implicit costs, they consider a case of proportional transaction costs. These proportional trading costs can be either explicit or implicit, whichever is greater. However, the study of transaction costs requires the identification of the type of cost to be estimated in order to explore effective ways of having a good estimate. Thus, in our study, we concentrate on implicit transaction costs as these are invisible and difficult to measure. These costs can turn high-quality investments into moderately profitable investments or low-quality investments into unprofitable investments (D'Hondt and Giraud, 2008).

Most models presented in the literature are static models: a decision is made, then not further modified. They are essentially single-period models, since there is only one decision to be made, for the first period. Real-life decision processes are more complicated. Although we must make an initial decision now, there will be many opportunities to adjust down the road. We do not, today, have a complete decision basis for future decisions – the future is unknown. Stochastic programming models hence are dynamic, covering multiple time periods with associated separate decisions, and they account for the stochastic decision process. The main features of stochastic programming are scenarios and stages. The uncertainty about future events are captured by a set of scenarios, which is a representative and comprehensive set of possible realisations of the future. Stochastic programming recognises that future decisions happen in stages. A first decision now, then after a certain time period, a second decision is made which depends upon the first stage decision and the events that occurred during the time period.

The purpose of this study is to construct a multi-stage stochastic MAD portfolio optimisation model with random transaction costs that captures assets' returns and risk due to uncertainty. The model employs stochastic programming with recourse by taking into account rebalancing of portfolio composition as the uncertainty of returns get realised. MAD models that are proposed in the literature do not take the uncertainty of the future into account. Of the models that account for transaction costs in portfolio selection, they do not consider random transaction costs. Konno and Wijayanayake (2001) proposed the deterministic MAD model with transaction costs modelled by a concave function. They use a linear cost function as an approximation to the concave cost function. Gulpinar et al. (2004) propose a multi-stage mean-variance portfolio analysis with non-random transaction costs. Yu et al. (2006) propose a multi-period portfolio selection with l_∞ model. They employ the l_∞ function to control the risk in every period. However, no transaction costs are considered in the model. Again, the model does not account for uncertainty of future events.

This paper is organised as follows. In Section 2, we discuss the formulation of the stochastic MAD model with random transaction costs for multi-period optimal portfolio selection. Transaction costs and portfolio rebalancing constraints are explained. Section 3 is devoted to discussing implicit transaction costs during trading and how they affect portfolio returns. Explicit transaction costs are not considered in this study as these can be known before execution of the trade. In Section 4, we demonstrate the application of the model to securities taken from the Johannesburg stock market, courtesy of INet Bridge. The merits of the model are clearly revealed in the application. We conclude, in Section 5, by giving a summary of our findings.

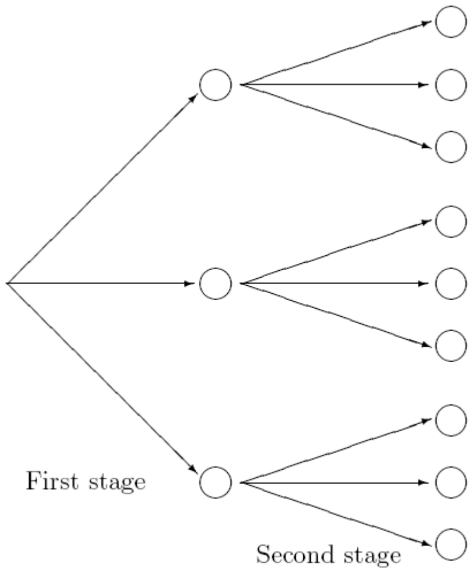
2 Problem statement

We determine a multi-period discrete-time optimal portfolio strategy over a given investment horizon. We shall achieve this by extending the MAD model to account for uncertainty in assets' returns and random transaction costs incurred during portfolio rebalancing. To define the problem, we divide the entire investment horizon T into two discrete time intervals T_1 and T_2, where $T_1 = 0, 1, \cdots, \tau$ and $T_2 = \tau + 1, \cdots, T$. The portfolio is structured over this period in terms of both assets' returns and risk, where the risk is measured by the MAD of assets' returns from expected portfolio return. Period τ defines the planning horizon. During T_1, an investor makes decisions and adjustments to his portfolio at each time-stage as some return realisations unfold. Thus, after the initial investment at $t = 0$, the portfolio may be restructured at discrete times $t = 1, 2, \cdots, \tau$; and after period τ, no further decisions are implemented until investment maturity at $t = T$. This restructuring of the portfolio brings with it transaction costs, as the investor sells or buys shares of some securities to balance his portfolio. A number of articles in the literature proposed models that do not account for these transaction costs. By ignoring such costs and considering only returns realised at time stages $t = 1, 2, \cdots, \tau$, we are overvaluing the portfolio or the portfolio may not be optimal.

2.1 Scenario generation

We consider a set of securities $I = \{i: i = 1, 2, \cdots, n\}$ for an investment. Let $R_t = \{R_1, \cdots, R_\tau\}$ be stochastic events at $t = 1, 2, \cdots, \tau$. The decision process is non-anticipative (i.e., a decision at a given stage does not depend on the future realisation of the random events). The decision at period t depends on the outcome at period $t - 1$. We define a scenario as a possible realisation of the stochastic variables $\{R_1, R_2, \cdots, R_\tau\}$. Hence, the set of scenarios corresponds to the set of paths followed from the root to the leaves of a tree, S_τ, and nodes of the tree at level $t \geq 1$ corresponds to possible realisations of R_t. Each node at a level t corresponds to a decision which must be determined at time t, and depends in general on R_t, the initial wealth of the portfolio and past decisions. Given the event history up to time t, R_t, the uncertainty in the next period is characterised by finitely many possible outcomes for the next observations R_{t+1}. The branching process is represented by a scenario tree. An example of a scenario tree with two time periods and three-three branching structure is shown in Figure 1.

Figure 1 Scenario tree



The uncertain return of the portfolio at the end of the period t is $R = R(x_t, r_t)$. This is a random variable with a distribution function, say F, given by

$$F(x, \mu) = p\{R(x, \tau) \leq \mu\}$$

We assume that the distribution F does not depend on the portfolio composition x. The expected return of the portfolio at the end of period t is

$$r_{pt} = E\left[R(x_t, R_t)\right] = r(x_t, R_t).$$

Suppose the uncertain returns of the assets, R_t, in period t are represented by a finite set of discrete scenarios $\Omega = \{s\colon s = 1, 2, \cdots, S\}$, whereby the returns under a particular scenario $s \in \Omega$ take the values $R_s = (R_{1s}, R_{2s}, \cdots, R_{ns})^T$ with associated probability $p_s > 0$, where $\sum_{s\in\Omega} p_s = 1.$ The mean return of assets in period t is

$$r_t = \sum\nolimits_{s \in \Omega} p_s R_{st}, \; t = 1, 2, \cdots, \tau.$$

The portfolio return under a particular realisation of asset return R_s of period t is denoted by $r_{st} = r(x_t, R_{st})$. The expected portfolio return of period t is now given by

$$r_{pt} = E\left[r\left(x_t, R_{st} \right) \right] = \sum\nolimits_{s \in \Omega} p_s r\left(x_t, R_{st} \right)$$

2.2 Capital allocation

As the problem is dynamic in nature, the wealth to be invested varies with time. Since we assume that the investor joins the market at $t = 0$, with an initial wealth W_0, W_t becomes the wealth at period t. The wealth can be re-allocated among the n-assets at the beginning of each of the τ consecutive time periods for portfolio rebalancing. The rate of asset' return of security i of period t in scenario s within the planning horizon is denoted by $R_i = (R_{1st}, R_{2st}, \cdots, R_{nst})$, where R_{ist} is the random return of security i of scenario s in period t. The return R_{st} has a mean defined by $r_{st} = E(R_{st})$, $t = 1, 2, \cdots, \tau$.

Let x_{ist} be the proportion of the risky asset i of scenario s at the beginning of period t. An investor is seeking the best investment strategy, $x_t = [x_{1st}, x_{2st}, \cdots, x_{nst}]$ for $t = 1, 2, \cdots, \tau$, such that

$$\sum\nolimits_{i=1}^{n} x_{it} = \sum\nolimits_{s=1}^{S} x_{ist} = 1, \; t = 1, 2, \cdots, \tau.$$

Let a_{ist} and v_{ist} be respectively the buying and selling volumes so that $x_{ist} + a_{ist} - v_{ist}$ is the volume invested in the i^{th} asset of scenario s at the beginning of period t. Thus, we also have

$$\sum\nolimits_{s=1}^{S} \left(x_{ist} + a_{ist} - v_{ist} \right) = 1, \; i = 1, 2, \cdots, n; \; t = 1, 2, \cdots, \tau.$$

Let k_{ist} and l_{ist} be, respectively, the rate of buying and selling transaction costs of the volume of asset i of scenario s bought or sold for portfolio rebalancing at the beginning of period t. It must be noted that the following are consequences of transaction costs:

a The fact that transactions have costs ensures that for the same asset, $a_{ist} \cdot v_{ist} = 0$; that is, the buy and sell variables corresponding to the same asset can never be non-zero simultaneously.

b The incorporation of transaction costs in the model provides essential 'friction' which without it the optimisation has complete freedom to re-allocate the portfolio every time period, which, (if implemented) can result in significantly poorer realised performance than forecast, due to excessive transaction costs.

Hakansson (1971) explains that in the absence of transaction costs, myopic policies are sufficient to achieve optimality. In our model, money can only be added at $t = 0$, and not in subsequent periods. Thus, only the initial wealth, W_0, is considered.

2.3 Transaction costs and balance constraints

In rebalancing the portfolio at any period $t > 0$, $t = 1, 2, \cdots, \tau$, buying and selling of securities take place. The buying and selling of securities during initial trading and rebalancing of the portfolio result in the investor incurring some explicit and implicit transaction costs. Explicit costs are directly observable, and they include market fees, clearing and settlement costs, brokerage commissions, and taxes and stamp duties. These costs do not rely on the trading strategy and can easily be determined before the execution of the trade. On the other hand, implicit costs are invisible. They depend mainly on the trade characteristics relative to the prevailing market conditions. They are strongly related to the trading strategy and provide opportunities to improve the quality of execution. In this study, we take transaction costs to mean implicit transaction costs, details of which are explained in Section 3. The impact of transaction costs on the mean-variance model have been studied by Konno and Yamazaki (1991), and Gulpinar et al. (2004). The decision made at period t depends on x_{ist} and the yield of the investment in asset i of scenario s as

$$r_{it} = r_{ist} \cdot p_s \cdot (x_{ist} + a_{ist} - v_{ist}), i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau. \quad (2.1)$$

and

$$0 \leq \sum_{i \in A} b_{ist} = \sum_{j=1, j \neq i} q_{jst}; i = 1, \cdots, n; s = 1, \cdots, S \quad (2.2)$$

where $b_{ist} = b_{ist}(a_{ist})$ is the amount of money used for buying volume a_{ist}, and $q_{ist} = q_{ist}(v_{ist})$ is the money obtained from selling volume v_{ist} of asset i of scenario s in period t. Note here that A is the set containing all assets i for which volumes have been bought. Thus, constraint (2) justifies that the amount of money used to buy volumes of asset i should be the same as the amount obtained from selling volumes of asset j, ($j \neq i$) of period t. Another constraint is given by

$$0 \leq v_{ist} \leq x_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau. \quad (2.3)$$

This constraint explains that the volume of asset i of scenario s in period t sold for portfolio rebalancing should not exceed the volume of the asset in the portfolio. The third constraint is the one discussed above, that is,

$$a_{ist} \cdot v_{ist} = 0 \quad (2.4)$$

It should be noted that the scenarios may reveal identical value for the uncertain quantities up to a certain period. These scenarios that share common information must yield the same decisions up to that period. Thus, we have the constraint

$$x_{ist} = x_{is't},$$

for all scenarios s and s' with identical past up to time t.

2.4 *Expected wealth*

Generally, the objective of an investor is to minimise portfolio risk while at the same time maximising expected portfolio return on investment, or achieving a prescribed expected return. Thus, the mean rate of return of the portfolio of period t is given by

$$r_{pt} = p_s \cdot r_{1st} \cdot (x_{1st} + a_{1st} - v_{1st}) + \cdots + p_s \cdot r_{nst} \cdot (x_{nst} + a_{nst} - v_{nst})$$
$$= \sum_{i=1}^{n} p_s \cdot r_{ist} \cdot (x_{ist} + a_{ist} - v_{ist}),\ s = 1, \cdots, S;\ t = 1, \cdots, \tau.$$

The wealth of period t, without transaction costs, is given by

$$W_t = (1 + r_{pt}) \cdot W_{t-1},\ t = 1, \cdots, \tau \tag{2.5}$$

Clearly, it can be said that the expected rate of return is a linear function of (x_{ist} + a_{ist} – v_{ist}). Taking transaction costs into consideration, we have the transaction cost of asset i in scenario s of period t to be

$$k_{ist} a_{ist} + l_{ist} v_{ist}$$

where $k_{ist}a_{ist}$ is the cost for buying volume a of asset i and $l_{ist}v_{ist}$ is the cost of selling volume v of asset i in period t. Since a_{ist} and v_{ist} cannot be non-zero simultaneously for each asset i (as is shown later in this section), we either have

$$k_{ist} a_{ist} = 0 \text{ or } l_{ist} v_{ist} = 0$$

or both being zero if there is no buying or selling of asset i in scenario s of period t. Allowing co-movement of assets' prices and corresponding implicit transaction costs during trading, we say that each scenario asset price is associated with it an implicit transaction cost. Thus, the expected transaction cost of asset i in period t is given by

$$\sum_{s=1}^{S} p_s \{k_{ist} a_{ist} + l_{ist} v_{ist}\},\ i = 1, \cdots, n;\ t = 1, \cdots, \tau.$$

We therefore observe that the total portfolio transaction cost of period t becomes

$$\sum_{i=1}^{n} \left[\sum_{s=1}^{S} p_s \{k_{ist} a_{ist} + l_{ist} v_{ist}\} \right],\ t = 1, 2, \cdots, \tau.$$

Denoting the net expected portfolio return of period t by N_{pt}, we get

$$N_{pt} = r_{pt} - \sum_{i=1}^{n} \left[\sum_{s=1}^{S} p_s \{k_{ist} a_{ist} + l_{ist} v_{ist}\} \right].$$

Thus, the wealth of period t taking transaction costs into account is given by

$$W_t = (1 + N_{pt}) \cdot W_{t-1},\ t = 1, \cdots, \tau. \tag{2.6}$$

2.5 Expected risk

The portfolio risk for any realisation of any period is measured by the MAD of the realised returns relative to the expected portfolio return, r_{pt}. Konno and Yamazaki (1991) develop the deterministic MAD model in an attempt to improve on the famous Markowitz (1952) mean-variance model. Their MAD model has portfolio risk expressed as

$$H(p) = \frac{1}{T} \sum_{t=1}^{T} \left| \sum_{i=1}^{n} (r_{it} - r_i) x_i \right|$$

where *rit* is the realised return of asset *i* of period *t*, r_i is the expected return of asset *i* per period, and x_i is the proportion of wealth invested in asset *i*.

We therefore extend this formulation of portfolio risk by constructing a model that takes into account uncertainty of asset returns and randomness of transaction costs in portfolio rebalancing. Using the deterministic model as our basis, three key stochastic framework elements are incorporated to formulate the stochastic MAD model. The first element considered is the concept of scenarios. Since the deterministic model represents one particular scenario, the inclusion of multiple scenarios in capturing uncertainty result in the increase in the number of variable parameters. Thus, uncertainty is represented by a set of distinct realisations $s \in \Omega$. We now consider the scenario parameter, along with other parameters of security and time period. Secondly, the stochastic MAD model allows for the implementation of recourse decisions as unfolding information on assets' returns get realised. The third element is the probabilistic feature of the stochastic framework which assigns probabilities to scenarios. The parameter p_s represents scenario probability. Scenarios may reveal identical value for the uncertain quantities up to a certain period. Scenarios that share common information must yield the same decisions up to that period.

The stochastic MAD also incorporates transaction costs incurred during portfolio rebalancing at each time period. Thus, we obtain the following portfolio risk:

$$H(r_{pt}) = \frac{1}{\tau}\sum_{t=1}^{\tau}\left|\sum_{s=1}^{S} p_s (R_{st} - r_{st}) x_{st}\right| \tag{2.7}$$

where R_{st}, r_{st} and x_{st} are as defined earlier. It should be noted that information revealed in the literature show transaction costs being approximated by a linear function or a step function. In some cases, they are considered to be proportional to volume of asset bought or sold. In either case, we argue that since returns are random, transaction costs can as well be random. Hence, in our model, we consider random transaction costs. Yu et al. (2003) considered symmetric transaction costs for buying and selling during portfolio rebalancing in their general mean-risk model they proposed. This makes computation easier although real-life situations may be more complex than this. Depending on the demand of the asset, transaction cost for buying asset *i* may be different from transaction cost for selling asset *j*, if asset *i* is on great demand and asset *j* is not.

2.6 Lower and upper bounds of variables

In portfolio optimisation, it is sometimes possible for an investor to sell an asset that one does not own. This is called short-selling or simply shorting. An investor borrows an asset which he then sells. Later, when the price of the asset falls on the market, the investor buys the asset and pays back to the lender. However, short-selling is only profitable if the asset price declines. It is very risky for investors as potential for loss is great. Thus, bounds U_{ist} on decision variables are put to prevent short-selling and enforce further restrictions imposed by the investor. We have the following constraints:

$$0 \le x_{ist} \le U_{ist},\ i = 1, \cdots, n;\ s = 1, \cdots, S;\ t = 1, \cdots, \tau. \tag{2.8}$$

and

$$0 \le v_{ist} \le x_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau. \tag{2.9}$$

Constraint (8) ensures diversification of the portfolio. It restricts the investor from investing all his wealth in one security, which can be very risky. Constraint (9) prohibits the selling of more of an asset *i* than his portfolio has.

2.7 Multi-stage stochastic MAD model

We express the multi-stage portfolio selection problem as a minimisation of portfolio risk subject to constraints describing the growth of the portfolio in all scenarios, a performance constraint, and bounds on the variables.

We constrain the final expected wealth to be a particular value α. The optimisation model intends to find the least risky investment strategy to achieve the expected specified wealth. Alternatively, we can achieve the same strategy by constraining the net expected return to be at least λ, or the gross expected portfolio return to be at least θ. Varying λ or θ and re-optimising generate a set of optimal portfolios, forming the efficient frontier. We now state the stochastic MAD model as follows:

Minimise

$$H(r_{pt}) = \frac{1}{\tau}\sum_{t=1}^{\tau}\left|\sum_{s=1}^{S} p_s (R_{ist} - r_t) x_{ist}\right| \tag{2.10}$$

subject to

$$\begin{aligned}
&N_{p\tau} \ge \lambda \\
&W_t = (1 + N_{pt}) W_{t-1}, t = 1, \cdots, \tau, \\
&0 \le \sum_{i \in A} b_{ist} \le \sum_{j=1, j \ne i}^{n} q_{ist}, s = 1, \cdots, S; t = 1, \cdots, \tau \\
&0 \le v_{ist} \le x_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\
&0 \le x_{ist} \le U_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\
&0 = a_{ist} \cdot v_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\
&b_{ist} \ge 0, q_{ist} \ge 0, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\
&x_{ist} = x_{is't}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau,
\end{aligned}$$

In addition to constraining the final rate of return of the portfolio, constrains of the form

$$N_{pt} \ge \lambda_t, t = 1, \cdots, \tau,$$

can be added to ensure any desired intermediate expected performance. We assume that no borrowing is done and the portfolio is self-financing. It deserves mention that the existence of a riskless asset among the securities is regarded as a special case in the stochastic MAD formulation (10) above.

It has been noted earlier that for each asset *i*, a_{ist} and v_{ist} cannot simultaneously be non-zero. We now prove the assertion.

Theorem 1:

Assume that there are transaction costs of model (10), the complementary constraint $a_{ist} \cdot v_{ist} = 0$ can be eliminated from the model.

Proof:

Without loss of generality, let us consider $a_t \cdot v_t = 0$. Let $(x_1^*, \cdots x_n^*; a_1^*, \cdots, a_n^*; v_1^*, \cdots, v_\tau^*)$ be an optimal solution of (10) without the complementary condition, and let us assume that $a_t \cdot v_t > 0$, $t \in M \subset \{1, \cdots, \tau\}$:

For $t \in M$, let

$$\begin{pmatrix} a_t' \\ v_t' \end{pmatrix} = \begin{pmatrix} a_t^* - v_t^* \\ 0 \end{pmatrix}$$

if $a_t^* \geq v_t^* \geq 0$ and

$$\begin{pmatrix} a_t' \\ v_t' \end{pmatrix} = \begin{pmatrix} 0 \\ v_t^* - a_t^* \end{pmatrix}$$

if $v_t^* \geq a_t^* \geq 0$.

Then $(x_1^*, \cdots, x_n^*; a_1', \cdots, a_\tau'; v_1', \cdots, v_\tau')$ satisfies all the constraints of (10). Also, it has the same objective as $(x_1^*, \cdots, x_n^*; a_1^*, \cdots, a_\tau^*; v_1^*, \cdots, v_\tau^*)$. This completes the proof.

Thus, we can now state problem (10) without the complementary constraint $a_{ist} \cdot v_{ist} = 0$. Let us denote $y_{st} = (R_{ist} - r_{it})x_{ist}$, $i = 1, \cdots, n$; $s = 1, \cdots, S$; $t = 1, \cdots, \tau$. Since $R_{ist} = R_{ist}(x_{ist} + a_{ist} - v_{ist})$ and $r_{it} = r_{it}(x_{ist} + a_{ist} - v_{ist})$, we have $y_{st} = y_{st}(x_{ist} + a_{ist} - v_{ist})$ as well. Thus, formulation (10) leads to the following minimisation problem:

Minimise

$$H(r_{pt}) = \frac{1}{\tau} \sum_{t=1}^{\tau} \left| \sum_{s=1}^{S} p_s y_{st} \right| \tag{2.11}$$

subject to

$$\begin{aligned} & N_{p\tau} \geq \lambda, \\ & W_t = (1 + N_{pt}) W_{t-1}, t = 1, \cdots, \tau, \\ & 0 \leq \sum_{i \in A} b_{ist} \leq \sum_{j=1, j \neq i}^{n} q_{ist}, s = 1, \cdots, S; t = 1, \cdots, \tau, \\ & 0 \leq v_{ist} \leq x_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\ & 0 \leq x_{ist} \leq U_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\ & b_{ist} \geq 0, q_{ist} \geq 0, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\ & x_{ist} = x_{is't}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau \end{aligned}$$

which is equivalent to the linear programme:

Minimise

$$H\left(r_{pt}\right)=\frac{1}{\tau}\sum_{t=1}^{\tau}Z_t \tag{2.12}$$

subject to

$$0\le Z_t+\sum_{s=1}^{S}p_s y_{st},\, i=1,\cdots,n;\, t=1,\cdots,\tau,$$
$$0\le Z_t-\sum_{s=1}^{S}p_s y_{st},\, i=1,\cdots,n;\, t=1,\cdots,\tau,$$
$$N_{p\tau}\ge\lambda,$$
$$W_t=\left(1+N_{pt}\right)W_{t-1},\, t=1,\cdots,\tau,$$
$$0\le\sum_{i\in A}b_{ist}\le\sum_{j=1,j\ne i}^{n}q_{ist},\, s=1,\cdots,S;\, t=1,\cdots,\tau,$$
$$0\le v_{ist}\le x_{ist},\, i=1,\cdots,n;\, s=1,\cdots,S;\, t=1,\cdots,\tau,$$
$$0\le x_{ist}\le U_{ist},\, i=1,\cdots,n;\, s=1,\cdots,S;\, t=1,\cdots,\tau,$$
$$b_{ist}\ge 0,\, q_{ist}\ge 0,\, i=1,\cdots,n;\, s=1,\cdots,S;\, t=1,\cdots\tau,$$
$$x_{ist}=x_{is't},\, i=1,\cdots,n;\, s=1,\cdots,S;\, t=1,\cdots,\tau,$$

where $Z_t\ge\left|\sum_{s=1}^{S}p_s y_{st}\right|$. The first two constraints ensure that the deviation is absolute.

3 Transaction cost measurement

From retail to more professional investors and practitioners, there is great concern on transaction costs as information gleaned from research reveal that lower transaction costs result in higher portfolio returns. Transaction costs include all costs associated with trading, which can be split into explicit and implicit costs.

Explicit costs are directly observable, and they include market fees, clearing and settlement costs, brokerage commissions, and taxes and stamp duties. These costs do not rely on the trading strategy and can easily be determined before the execution of the trade. On the other hand, implicit costs are invisible. These can broadly be put into three categories, namely market impact, opportunity costs, and spread. These costs can turn high-quality investments into moderately profitable investments or low-quality investments into unprofitable investments (D'Hondt and Giraud, 2008).

To provide investors with competitive portfolio returns, investment managers must manage transaction costs proactively. Perhaps the reason why managers find themselves in a difficult situation is that these implicit costs are 'hidden' in the stock price. They depend mainly on the trade characteristics relative to the prevailing market conditions. They are strongly related to the trading strategy and, as variable costs, provide opportunities to improve the quality of execution. Missed trading opportunity costs are incurred when investors fail to fulfil their orders. Price movements or unavailability of the security (lack of liquidity) can be cause of such investor behaviour.

The contribution of opportunity costs to total implicit costs cannot be underestimated. The effects of market impact costs and opportunity costs are inversely proportional. Minimising market impact and reducing opportunity costs form a set of conflicting objectives. When an investment decision is immediately executed without delay, implicit costs are largely a result of market impact or liquidity restrictions only, and defined as the deviation of the transaction price from the 'unperturbed price' that would have prevailed if the trade had not occurred. Thus, market impact can be explained as the additional costs incurred by the investor for immediate trade execution. This is the resultant change that occurs when the number of shares of stocks an investor wants to buy or sell exceeds the number other market participants are willing to buy or sell at that price. Although market impact cost tends to decrease with time, such a delay in execution would result in increasing opportunity costs. This gives rise to what is called the 'trader's dilemma'.

In this study, we assume immediate trade execution, thus taking market impact to account for the total implicit costs. We shall use the spread mid-point benchmark and the transaction price. The transaction price shall be the last price of the month. We follow the implicit transaction calculation as given by Hau (2006). We calculate the effective spread as twice the distance from the midprice measured in basis points. For a transaction price P^T and the mid-price P^M, mid-point of the bid-ask spread, we obtain the effective spread (implicit transaction cost) as

$$SPREAD^{Trade} = 200 \times \frac{|P^T - P^M|}{P^M}.$$

We take the last price as the transaction price for every stock considered, and for each month. These are prices of assets traded on the Johannesburg Stock Exchange from 1 January 2008 to 30 September 2012. These historical data have been obtained courtesy of INet Bridge.

4 Model application and analysis

In applying the model to analyse the historical data, we use GAMS software. The 13 assets used in the analysis are selected from securities in the Johannesburg stock market. Selection is on the basis of mean asset returns over the entire duration considered, and those assets having the highest mean returns are chosen to comprise the initial portfolio. Since we are using historical data, we consider empirical distributions computed from past returns as equiprobable scenarios. Observations of returns over, say N, overlapping periods of length δ_t are considered as the N_s possible outcomes (or scenarios) of the future returns and a probability of $\frac{1}{N_s}$ is assigned to each of them.

We assume that we have historical prices at each of the t periods, $t = 1, \cdots, \tau$, of stocks under consideration. For each period t, we compute the realised vector over the previous period, say 1 month, which is further considered as one of the Ns scenarios for the future returns on the assets. Thus, for example, a scenario R_{ist} for the return on asset i is obtained as

$$R_{i,s,t} = \frac{V_{i,t} - V_{i,t-1}}{V_{i,t-1}}$$

where

- R_{ist} is the rate of return of asset i of scenario s of period t
- $V_{i,t-1}$ is the closing price per share of asset i in period $t-1$
- $V_{i,t}$ is the closing price per share of asset i of period t.

On the basis of selecting assets with the best mean returns for the period 1 January 2008 to 30 September 2012, the 13 securities heading columns in Table 1 were chosen to comprise our initial portfolio. The assets mean returns are denoted by $R_1, R_2, \cdots, R_{13}$.

4.1 Scenario generation for stage 1

We consider five scenarios in our demonstration and apply the model over two stages. This consideration is taken noting that in stochastic programming the scenario tree grows exponentially. We take the empirical distributions of the 13 securities comprising our initial portfolio. Since for each security we have 54 monthly returns, we number the months from 1 to 54 and use random numbers to select asset returns corresponding to a scenario of a security. As explained above, we consider implicit transaction costs, and in particular, the market impact costs, obtained from the effective bid-ask spread, the calculation of which is also given in the above section. The effective bid-ask spread corresponding to each selected asset return for each scenario is used to calculate the market impact cost corresponding to the asset's return in that scenario. We consider these transaction costs as random since they are randomly selected together with corresponding returns.

4.2 End-of-first-stage portfolio selection

A GAMS software is used to execute the model. In identifying assets that should comprise the second stage portfolio, a condition is given in the model to show an expected value of each asset and assets with positive and higher expected values are chosen. It should be noted that each asset has five scenarios and each scenario is equally likely to occur and is given a probability of $\frac{1}{5}$. The portfolio return rate at this stage is obtained by dividing the sum of the assets' expected return rates by the number of assets in the optimal portfolio. The implicit cost for each of the 13 assets is taken into account as it is in the buying of each asset that the cost is incurred. The transaction cost is given as a rate.

4.3 End-of-first-stage portfolio rebalancing

At the end of the first stage, an investor decides on his first-stage optimal portfolio as given by the investor's chosen diversification limit, the gross portfolio mean return or the net portfolio mean return as the case may be, and the associated risk given by the MAD. It should be noted that the new monetary values of the assets become known after running the model for the second stage. As in the first stage, the securities from the first-stage optimal portfolio are each having five scenarios, each with a probability of occurring of $\frac{1}{5}$. Each scenario has, associated with it, a cost rate (buying or selling cost rate). Each cost rate also has a probability of $\frac{1}{5}$. After running the model, the new proportions of assets become known. This happens concurrently with the calculation of transaction cost of an asset when volume is bought or sold. These transaction costs are expected costs of the volume of an asset bought or sold. The product of asset proportion, asset cost rate and the associated probability gives the expected transaction cost of an asset during portfolio rebalancing. These transaction costs are known when the investor has decided on his or her optimal portfolio at the end of stage 2.

4.4 Analysis of results

4.4.1 Stage 1

We consider an investor who has R10000 to spend on his initial portfolio. We find that as we diversify the investor's portfolio by decreasing the diversification limit from 0.4 to 0.1, the gross mean portfolio return remains fixed at 0.013 and the risk is very small in each case. However, the net mean portfolio return fluctuates between 0.008 and 0.011, reflecting fluctuating transaction costs. Table 1 has this information where the phrase 'div. lim' means 'diversification limit'.

Table 1 First stage optimal portfolios

Div. lim	*Gross mean*	*No. of assets*	*Net mean*	*MAD*	*Wealth*
0.100	0.013	11	0.011	2.1684×10^{-19}	10,114.320
0.125	0.013	12	0.011	2.40551×10^{-18}	10,112.736
0.150	0.013	10	0.011	1.5992×10^{-18}	10,111.780
0.175	0.013	10	0.011	2.6766×10^{-19}	10,107.344
0.200	0.013	6	0.008	4.3368×10^{-19}	10,081.526
0.225	0.013	6	0.008	1.0842×10^{-19}	10,081.891
0.250	0.013	8	0.009	2.3039×10^{-19}	10,085.628
0.275	0.013	10	0.008	0	10,081.730
0.300	0.013	7	0.011	2.50722×10^{-19}	10,105.436
0.325	0.013	6	0.010	7.2844×10^{-20}	10,098.718
0.350	0.013	7	0.011	1.4×10^{-18}	10,106.596
0.375	0.013	4	0.011	2.710×10^{-19}	10,108.360
0.400	0.013	4	0.011	2.168×10^{-19}	10,108.360

We then fix each of the diversification limits and let the gross mean portfolio return increase from 0.013 to values when the portfolio becomes 'saturated', i.e., no more buying and selling of assets. Thus, creating a set of optimal portfolios for each diversification limit considered. We note that, regardless of the diversification limit chosen, the net mean portfolio return for each chosen gross mean portfolio return is the same for all diversification limits considered. Thus, transaction costs are the same for each gross mean portfolio return selected. Table A3 in Appendix has this information. It is also evident that even the risk assumes the same values for each gross mean portfolio return considered for all diversification limits, except at the portfolio saturation point. Thus, the efficient frontiers at the various diversification limits do not differ much.

However, we observe that the maximum wealth is achieved at the optimal portfolio saturation point for each diversification limit. This is also where the risk is highest. Efficient frontiers of net mean portfolio returns and gross mean portfolio returns reveal the impact of neglecting transaction costs in portfolio selection. The gap between the two frontiers is the exaggeration that results. It is therefore up to each investor to choose the wealth-risk-diversification limit combination he or she desires.

4.4.2 Stage 2

As in stage 1, we allow the diversification limit to vary from 0.2 to 0.4 as shown in Table 2. We consider a first-stage optimal portfolio with five securities. We find that optimal portfolios in stage 2 have the same gross mean portfolio return and the same net mean portfolio return, regardless of the diversification limit. Here, we note that transaction costs for the optimal portfolios are the same. It is also observed that the risk assumes the same value at each diversification limit. Small variations in the expected wealth of optimal portfolios is due to rounding error of the net mean portfolio return during execution of the programme. Similar results are observed with all stage 1 optimal portfolios.

Table 2 Second stage optimal portfolios

Div. lim	*Gross mean*	*No. of assets*	*Net mean*	*MAD(risk)*	*Wealth*
0.200	0.007	5	0.006	0.013	10,423.815
0.225	0.007	5	0.006	0.013	10,424.277
0.250	0.007	5	0.006	0.013	10,425.228
0.275	0.007	5	0.006	0.013	10,424.843
0.300	0.007	5	0.006	0.013	10,423.679
0.325	0.007	5	0.006	0.013	10,424.327
0.350	0.007	5	0.006	0.013	10,424.321
0.375	0.007	5	0.006	0.013	10,424.290
0.400	0.007	5	0.006	0.013	10,424.259

It is evident from the table findings that each optimal portfolio at each diversification limit considered, incurs an overall implicit transaction cost of $\frac{1}{7}$ of the returns on investment. Thus, implicit transaction costs amount to approximately 14.3% of the fund invested during initial trading.

We again find optimal portfolios at each diversification limit, the results of which are shown in Table A4 in Appendix. For each gross mean portfolio return, the net mean portfolio returns are the same and also equal are the associated risks. Unlike in stage 1, expected portfolio wealth is almost invariant for each pair of gross mean portfolio return and risk, and for each diversification limit. Thus, efficient frontiers are almost the same regardless of diversification limit.

4.4.3 Comparison of stage 1 and stage 2 optimal portfolios

The in-sample analysis explains the advantages of portfolio rebalancing as depicted from the efficient frontiers of two sets of optimal portfolios in which diversification limit is the same. Stage 2 portfolios are superior, hence the need for portfolio rebalancing. It is again clear that the transaction costs incurred when buying initial portfolio and during portfolio rebalancing have a bearable impact on portfolio returns. Hence, ignoring transaction costs results in exaggerated optimal portfolios.

4.5 Conclusions

In this study, we propose a multi-stage stochastic MAD model with random transaction costs in optimal portfolio selection. We view our contributions to include:

1 the development of a strategy that captures uncertainty in stock prices and in corresponding implicit trading costs by way of scenarios

2 the development of a stochastic MAD model that optimises portfolios in the presence of random implicit transaction costs, and which captures risk due to uncertainty.

The methodology allows investors and investment managers to choose optimal portfolios realising the impact of associated implicit transaction costs. It is a LP model, and hence reduces considerably the time needed to reach a solution. It is therefore feasible for large-scale portfolio selection. It is left for future research to have a model that takes into account uncertainty of both stock prices and implicit transaction costs as well as explicit trading costs. The study also has some limitations. The calculation of asset rate-of-return using the asset's closing price of the month may not be the most accurate measure of asset's monthly rate-of-return. However, since all assets' rates-of-returns are obtained in the same way, this does not prejudice the findings.

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Appendix

Table A1 (a) Mean asset returns

Month	*AVI*	*ASR*	*APN*	*CSB*	*CLS*	*CML*	*MPC*
1	0.099	0.422	–0.028	0.067	0.124	0.127	0.045
2	–0.102	–0.036	0.036	0.041	–0.057	–0.074	–0.078
3	–0.051	0.297	–0.027	0.143	–0.017	0.056	–0.082
4	–0.027	0.038	0.057	–0.021	0.014	–0.152	–0.042
5	–0.112	–0.036	–0.031	–0.138	–0.122	–0.188	–0.063
6	0.093	–0.125	0.32	0.036	0.062	0.099	0.304
7	0.084	0.071	0.113	0	0.179	0.04	0.134
8	–0.026	–0.193	–0.091	0.1	0	0.106	0.003
9	0.029	–0.173	–0.2	–0.018	0.03	–0.13	0.079
10	0.225	–0.2	0.101	–0.074	–0.048	0	–0.003
11	0.116	0.213	–0.103	0.04	0.118	–0.076	0.034
12	–0.074	–0.186	0.26	0.096	–0.029	–0.026	0.051
13	0.026	–0.139	–0.02	–0.035	–0.106	–0.022	–0.085
14	–0.136	0.294	–0.099	0.236	0.017	0.114	0.019
15	–0.046	0.02	–0.053	–0.043	0.097	0.163	0.076
16	0.061	0.069	0.111	0.022	0.064	–0.075	0.04
17	–0.029	0	0.141	–0.038	0.049	0.159	0.035
18	0.117	0.104	0.087	0.078	0.116	0.113	0.087
19	0.001	0.102	–0.017	0.014	–0.009	0.118	0
20	0	0.079	0.057	0.081	0.128	0.013	0.114
21	0.097	0.029	0.07	-0.077	0.079	0.048	0.056
22	–0.074	0.08	0.027	0.017	0.018	0.035	–0.092
23	0.083	0.004	0.082	0.056	0.083	0.048	0.074
24	0.012	–0.011	–0.084	–0.027	–0.028	0.017	0.005
25	0.033	0.036	0.053	–0.014	0.042	–0.006	0.095
26	0.089	0.069	0.12	0.034	0.074	0.101	0.032
27	0.008	0.018	0.05	–0.011	0.044	0.176	0.053
28	–0.031	–0.133	–0.052	–0.05	0.051	–0.07	0.069
29	–0.049	0.015	–0.039	0.072	0.043	0.028	–0.001
30	0.131	0.112	0.071	–0.027	0.065	0.16	0.128
31	0	–0.091	0.032	–0.068	0.037	0.004	–0.022
32	0.088	0.117	0.118	0.147	0.179	0.16	0.113
33	0.029	0.077	–0.005	0.212	0.034	0.014	0.157
34	0	0	–0.005	–0.049	–0.049	0.125	0.009
35	0.076	0.125	–0.011	0.057	–0.003	0.111	0.036
36	–0.005	0.055	–0.074	–0.048	–0.085	–0.069	–0.145

Table A1 (a) Mean asset returns (continued)

Month	*AVI*	*ASR*	*APN*	*CSB*	*CLS*	*CML*	*MPC*
37	0.01	0.028	–0.043	–0.074	–0.029	–0.052	0.054
38	–0.01	0.056	–0.031	0.116	0.105	0.076	0.022
39	0.027	–0.012	0.025	–0.005	0.031	0.067	0.097
40	–0.028	0.012	0.077	0.043	–0.025	0.034	–0.047
41	0.042	0.014	–0.037	–0.023	–0.013	–0.018	0.065
42	0.033	0.02	–0.008	–0.035	–0.053	0.037	0.081
43	0.014	–0.018	0.014	0.011	0.043	–0.006	0.001
44	–0.011	–0.105	0.08	0.113	–0.097	0.005	–0.088
45	0.101	0.099	0.044	0.001	0.107	0.128	0.135
46	0.049	0.012	0.023	0.086	0.068	0.007	0.047
47	0.063	–0.041	–0.008	0.055	0.039	0	–0.003
48	0.045	0.086	0.031	–0.005	–0.141	0.117	0.081
49	0.057	0.106	0.09	–0.053	0.106	0.083	0.034
50	0.057	–0.043	0.091	0.133	0.019	0.038	0.057
51	0.039	0.098	0.061	0.04	0.046	0.025	0.115
52	0.01	0.057	–0.065	–0.007	0	–0.099	–0.01
53	0.027	0.068	0.07	0.054	0.149	0.051	0.074
54	0.182	–0.01	0.153	0.102	0.016	0.064	0.101
55	–0.009	0.007	–0.008	0.053	–0.003	0.02	0.049

Table A1 (b) Mean asset returns

Month	*PNC*	*SPP*	*TRU*	*CPI*	*IPL*	*WHL*
1	0.136	–0.008	0.02	0.177	0.053	0.057
2	0	–0.025	–0.038	–0.064	–0.009	–0.025
3	–0.022	0.155	0.004	–0.096	–0.034	–0.013
4	–0.068	–0.071	–0.105	–0.076	–0.308	–0.054
5	–0.146	–0.048	0.002	–0.115	–0.005	–0.097
6	–0.057	0.051	0.267	0.107	–0.165	0.155
7	0.288	0.038	0.073	–0.013	0.226	0.067
8	–0.141	–0.065	–0.064	0.085	0.094	–0.069
9	–0.285	0.129	0.136	–0.156	–0.07	–0.034
10	–0.272	–0.056	–0.043	0.037	–0.084	0.084
11	–0.042	0.06	0.075	0.036	0.184	0.038
12	0.049	–0.035	0.033	0.034	–0.137	0.058
13	–0.063	–0.031	–0.117	0	–0.138	–0.115
14	0.039	–0.026	0.032	0.081	0.207	–0.042
15	0.022	0.026	0.058	0.202	0.02	0.053
16	0.089	0.07	0.068	0.051	0.101	0.042

Table A1 (b) Mean asset returns (continued)

Month	*PNC*	*SPP*	*TRU*	*CPI*	*IPL*	*WHL*
17	0.097	–0.004	0.019	0.035	–0.028	0.036
18	0	0.054	0.081	0.131	0.167	0.202
19	–0.004	0.043	–0.016	0.146	0.09	0.012
20	0.456	0.037	0.08	0.109	0.077	0.025
21	–0.033	0.066	0.059	0.049	0.02	0.083
22	–0.041	–0.041	–0.075	0.109	–0.006	–0.048
23	–0.02	0.074	0.048	0.111	0.092	0.072
24	0.1	–0.001	–0.028	–0.024	–0.092	0.024
25	0.125	0.028	0.166	0.065	0.17	0.127
26	0.078	0.021	0.058	0.153	0.07	0.093
27	–0.023	0.029	0.017	0.076	–0.018	0.041
28	0.103	0.019	0.022	–0.03	–0.025	–0.013
29	–0.002	0.022	–0.013	0.197	–0.107	0.035
30	0.131	0.071	0.087	0.093	0.124	0.085
31	–0.021	–0.017	0.004	0.035	0.059	–0.053
32	0.114	0.105	0.196	0.116	0.111	0.099
33	0.068	0.025	–0.014	–0.021	0.007	0.015
34	0.092	0.062	0.043	0.019	0.072	–0.038
35	0.107	–0.033	–0.005	0.156	0.041	0.02
36	–0.034	–0.071	–0.11	–0.109	–0.137	–0.128
37	0.03	0.046	0	0.039	0.021	0.136
38	0.029	0	0.106	0.057	0.016	0.047
39	0.014	0.026	0.078	0.037	0.033	0.072
40	0.099	–0.05	–0.047	0.085	–0.013	–0.007
41	0.154	–0.026	0.012	–0.042	0.041	0
42	0.022	0.02	–0.012	0.001	–0.052	0.061
43	–0.076	0.035	0.071	0.04	0.007	0.155
44	0.116	0.011	–0.091	0.019	–0.091	–0.04
45	0.138	0.007	0.138	–0.052	0.116	0.154
46	–0.029	0.16	–0.012	0.011	0.007	0.001
47	0.044	–0.035	–0.068	–0.033	0.044	–0.035
48	0.185	0.02	0.059	0.024	0.122	0.077
49	0.059	0.041	0.034	0.015	0.063	0.074
50	0.039	0.004	–0.001	0.108	0.052	0.067
51	0.113	0.05	0.028	0.083	0.089	0.01
52	–0.037	–0.097	–0.009	0.017	–0.021	0.003
53	0.035	0.035	0.089	–0.058	0.04	0.031
54	0.002	0.054	0.154	0.026	0.1	0.071
55	0.061	0.011	0.003	–0.012	0.005	0.016

Table A2 (a) Asset transaction cost rates

Month	*AVI*	*ASR*	*APN*	*CSB*	*CLS*	*CML*	*MPC*
1	0.0030	0.0023	0.0028	0.0036	0.0050	0.0119	0.0073
2	0.0004	0.0009	0.0028	0.0029	0.0023	0.0061	0.0051
3	0.0054	0.0008	0.0022	0.0052	0.0032	0.0425	0.0088
4	0.0087	0.0064	0.0121	0.0040	0.0107	0.0011	0.0019
5	0.0043	0.0062	0.0027	0.0080	0.0034	0.0004	0.0229
6	0.0000	0.0140	0.0072	0.0386	0.0098	0.0067	0.0012
7	0.0066	0.0048	0.0195	0.0214	0.0147	0.0018	0.0128
8	0.0048	0.0044	0.0306	0.0171	0.0548	0.0137	0.0024
9	0.0042	0.0061	0.0063	0.0078	0.0132	0.0018	0.0007
10	0.0100	0.0220	0.0027	0.0018	0.0213	0.0025	0.0060
11	0.0060	0.0003	0.0112	0.0258	0.0041	0.0085	0.0136
12	0.0143	0.0045	0.0170	0.0049	0.0015	0.0258	0.0019
13	0.0003	0.0126	0.0010	0.0055	0.0083	0.0135	0.0037
14	0.0033	0.0051	0.0029	0.0041	0.0019	0.0015	0.0020
15	0.0064	0.0023	0.0004	0.0001	0.0228	0.0026	0.0082
16	0.0057	0.0114	0.0042	0.0054	0.0092	0.0139	0.0140
17	0.0010	0.0087	0.0435	0.0165	0.0099	0.0073	0.0105
18	0.0537	0.0047	0.0067	0.0212	0.0045	0.0109	0.0028
19	0.0007	0.0052	0.0027	0.0474	0.0056	0.0053	0.0113
20	0.0060	0.0117	0.0126	0.0856	0.0046	0.0295	0.0033
21	0.0061	0.0303	0.0052	0.0532	0.0169	0.0033	0.0127
22	0.0070	0.0003	0.0087	0.0194	0.0268	0.0149	0.0049
23	0.0076	0.0029	0.0046	0.0074	0.0003	0.0008	0.0103
24	0.0056	0.0473	0.0025	0.0069	0.0127	0.0055	0.0026
25	0.0018	0.0058	0.0095	0.0070	0.0079	0.0046	0.0056
26	0.0149	0.0074	0.0327	0.0072	0.0009	0.0019	0.0045
27	0.0037	0.0208	0.0032	0.0233	0.0104	0.0035	0.0165
28	0.0202	0.0065	0.0256	0.0301	0.0051	0.0021	0.0073
29	0.0068	0.0225	0.0141	0.0061	0.0018	0.0056	0.0116
30	0.0298	0.0000	0.0343	0.0068	0.0695	0.0253	0.0094
31	0.0072	0.0197	0.0027	0.0408	0.0084	2.0000	0.0115
32	0.0157	0.0153	0.0066	0.0679	0.0040	0.0749	0.0058
33	0.0139	0.0238	0.0015	0.0019	0.0040	0.0286	0.0111
34	0.0142	0.0024	0.0079	0.0073	0.0013	0.0105	0.0024
35	0.0074	0.0186	0.0068	0.0326	0.0198	0.0066	0.0114
36	0.0316	2.0000	0.0082	0.0146	0.0487	0.0074	0.0133
37	0.0041	0.0083	0.0433	0.0155	0.0369	0.0082	0.0075
38	0.0459	2.0000	0.0058	0.0014	0.0029	0.0354	0.0033

Table A2 (a) Asset transaction cost rates (continued)

Month	*AVI*	*ASR*	*APN*	*CSB*	*CLS*	*CML*	*MPC*
39	0.0006	0.0386	0.0012	0.0046	0.0334	0.0177	0.0339
40	0.0743	0.0225	0.0097	0.0044	0.0148	0.0202	0.0543
41	0.0210	0.0290	0.0103	0.0952	0.0144	0.1277	0.0402
42	0.0208	0.0022	0.0179	0.0222	0.0308	0.1053	0.0089
43	0.0217	2.0000	0.0177	0.0396	0.0238	2.0000	0.0052
44	0.0053	2.0000	0.0151	0.0619	0.0033	0.0942	0.0112
45	0.0124	0.0942	0.0077	0.0759	0.0183	0.0408	0.0083
46	0.0086	0.0083	0.0800	0.0352	0.0215	0.0258	0.0014
47	0.0169	0.0645	0.0026	0.0829	0.0064	0.0077	0.0271
48	0.0060	0.0072	0.0185	0.0769	0.0068	0.0198	0.0020
49	0.0209	2.0000	0.0255	0.1452	0.0000	0.0217	0.0047
50	0.0330	0.0108	0.0132	0.1125	0.0043	0.0089	0.0277
51	0.0040	0.0189	0.0016	0.0894	0.0086	0.0308	0.0114
52	0.0032	0.0028	0.0110	2.0000	0.0078	0.0048	0.0164
53	0.0317	0.0237	0.0026	0.0734	0.0007	0.0543	0.0101
54	0.0180	0.0220	0.0085	0.0112	0.0104	0.0151	0.0228

Table A2 (b) Asset transaction cost rates

Month	*PNC*	*SPP*	*TRU*	*CPI*	*IPL*	*WHL*
1	0.0319	0.0019	0.0009	0.0063	0.0058	0.0062
2	0.0323	0.0072	0.0033	0.0105	0.0132	0.0010
3	0.0038	0.0044	0.0241	0.0061	0.0067	0.0018
4	0.0049	0.0035	0.0028	0.0411	0.0040	0.0029
5	0.0058	0.0021	0.0031	0.0031	0.0005	0.0013
6	0.0046	0.0024	0.0056	0.0031	0.0050	0.0010
7	0.0315	0.0029	0.0066	0.0056	0.0050	0.0008
8	0.0086	0.0946	0.0540	0.0121	0.0542	0.0069
9	0.0263	0.0009	0.0068	0.0003	0.0019	0.0027
10	0.0148	0.0030	0.0128	0.0108	0.0052	0.0023
11	0.0202	0.0019	0.0063	0.0459	0.0121	0.0689
12	0.0043	0.0109	0.0029	0.0071	0.0078	0.0099
13	0.0067	0.0007	0.0023	0.0028	0.0042	0.0044
14	0.0127	0.0017	0.0033	0.0032	0.0043	0.0057
15	0.0197	0.0041	0.0036	0.0191	0.0084	0.0020
16	0.0072	0.0012	0.0177	0.0047	0.0100	0.0050
17	0.0074	0.0001	0.0185	0.0050	0.0008	0.0004
18	0.0122	0.0085	0.0077	0.0005	0.0002	0.0098
19	0.0088	0.0007	0.0015	0.0054	0.0085	0.0015

Table A2 (b) Asset transaction cost rates (continued)

Month	*PNC*	*SPP*	*TRU*	*CPI*	*IPL*	*WHL*
20	0.0081	0.0078	0.0324	0.0030	0.0027	0.0046
21	0.0179	0.0085	0.0022	0.0031	0.0037	0.0271
22	0.0228	0.0022	0.0011	0.0101	0.0097	0.0234
23	0.0106	0.0305	0.0077	0.0143	0.0010	0.0220
24	0.0293	0.0053	0.0024	0.0039	0.0137	0.0073
25	0.0212	0.0050	0.0127	0.0125	0.0205	0.0421
26	0.0452	0.0019	0.0145	0.0145	0.0034	0.0121
27	0.0178	0.0024	0.0137	0.0069	0.0047	0.0223
28	0.0203	0.0093	0.0027	0.0053	0.0064	0.0071
29	0.0526	0.0043	0.0026	0.0037	0.0040	0.0019
30	0.0595	0.0113	0.0095	0.0065	0.0133	0.0016
31	2.0000	0.0058	0.0069	0.0127	0.0096	0.0028
32	0.0470	0.0020	0.0021	0.0140	0.0086	0.0300
33	0.0287	0.0151	0.0018	0.0047	0.0047	0.0114
34	0.0030	0.0023	0.0068	0.0149	0.0068	0.0025
35	0.0545	0.0032	0.0261	0.0091	0.0125	0.0184
36	0.0590	0.0017	0.0119	0.0092	0.0103	0.0032
37	0.0874	0.0142	0.0117	0.0012	0.0022	0.0117
38	0.0597	0.0039	0.0055	0.0241	0.0013	0.0080
39	0.2246	0.0186	0.0064	0.0102	0.0215	0.0058
40	0.0488	0.0107	0.0041	0.0015	0.0079	0.0302
41	0.0845	0.0252	0.1433	0.0481	0.0299	0.1060
42	0.0207	0.0077	0.0088	0.0236	0.0164	0.0105
43	0.2118	0.0357	0.0239	2.0000	0.0077	0.0008
44	0.0267	0.0080	0.0066	0.0142	0.0078	0.0123
45	0.2449	0.0741	0.0075	0.0714	0.0082	0.0370
46	0.2290	0.0050	0.0187	0.0546	0.0328	0.0481
47	0.0047	0.0093	0.0123	0.0165	0.0110	0.0169
48	0.0625	0.0012	0.0007	0.0479	0.0099	0.0273
49	0.1374	0.0228	0.0110	0.0147	0.0042	0.0069
50	0.0247	0.0179	0.0013	0.0165	0.0024	0.0608
51	0.0114	0.0011	0.0155	0.0469	0.0007	0.0109
52	0.0270	0.0062	0.0650	0.0408	0.0336	0.0198
53	0.0110	0.0082	0.0633	0.0524	0.0507	0.0112
54	0.0228	0.0445	0.0136	0.0202	0.0033	0.0347

Table A3 Stage 1 efficient frontiers at various diversification limits

Div. lim	*Gross mean*	*No. of assets*	*Net mean*	*MAD (risk)*	*Wealth*
0.1	0.013	11	0.011	2.1684×10^{-19}	10,114.320
	0.02	12	0.018	0.007	10,178.771
	0.03	11	0.025	0.017	10,253.167
	0.04	10	0.036	0.018	10,355.180
0.125	0.013	12	0.011	2.4×10^{-18}	10,112.736
	0.02	10	0.018	0.007	10,181.466
	0.03	10	0.028	0.017	10,276.063
	0.04	8	0.034	0.021	10,344.850
0.15	0.013	10	0.011	1.599×10^{-18}	10,111.780
	0.02	11	0.018	0.007	10,182.074
	0.03	11	0.025	0.017	10,248.470
	0.04	7	0.034	0.023	10,336.510
0.2	0.013	6	0.008	4.33×10^{-19}	10,081.526
	0.02	6	0.015	0.007	10,151.170
	0.03	7	0.026	0.017	10,255.143
	0.04	5	0.036	0.026	10,363.520
0.25	0.013	8	0.009	2.3×10^{-19}	10,085.628
	0.02	10	0.016	0.007	10,158.532
	0.03	7	0.026	0.017	10,256.967
	0.04	6	0.037	0.027	10,366.508
0.3	0.013	7	0.011	2.5×10^{-19}	10,105.436
	0.02	12	0.018	0.007	10,175.404
	0.03	7	0.028	0.017	10,282.641
	0.04	6	0.037	0.027	10,369.217

Table A4 Stage 2 efficient frontiers at various diversification limits

Div. lim	*Gross mean*	*No. of assets*	*Net mean*	*MAD (risk)*	*Wealth*
0.2	0.007	5	0.006	0.013	10,423.815
	0.010	5	0.009	0.015	10,460.266
	0.015	5	0.014	0.017	10,513.668
	0.020	5	0.019	0.020	10,562.684
	0.025	5	0.024	0.022	10,611.495
	0.030	5	0.029	0.025	10,664.308
	0.035	5	0.034	0.027	10,714.813
0.225	0.007	5	0.006	0.013	10,424.277
	0.010	5	0.009	0.015	10,459.822
	0.015	5	0.014	0.017	10,513.566
	0.020	5	0.019	0.020	10,562.732
	0.025	5	0.024	0.022	10,611.730
	0.030	5	0.029	0.025	10,662.818
	0.035	5	0.034	0.027	10,715.476

Table A4 Stage 2 efficient frontiers at various diversification limits (continued)

Div. lim	*Gross mean*	*No. of assets*	*Net mean*	*MAD (risk)*	*Wealth*
0.25	0.007	5	0.006	0.013	10,425.228
	0.010	5	0.009	0.015	10,461.394
	0.015	5	0.014	0.017	10,513.540
	0.020	5	0.019	0.020	10,562.598
	0.025	5	0.024	0.022	10,611.730
	0.030	5	0.029	0.025	10,660.993
	0.035	5	0.034	0.027	10,714.008
0.275	0.007	5	0.006	0.013	10,424.843
	0.010	5	0.009	0.015	10,460.696
	0.015	5	0.014	0.017	10,513.540
	0.020	5	0.019	0.020	10,562.598
	0.025	5	0.024	0.022	10,611.730
	0.030	5	0.029	0.025	10,660.733
	0.035	5	0.034	0.027	10,712.199