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Potential Gaps during the Transition from the Embodied through Symbolic to Formal Worlds of Reflective Symmetry

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Even though reflective symmetry is heavily embedded in the everyday, learners continue to experience challenges when they mathematize concepts from this informal/everyday context. In this article we argue that symmetry exists in nature, it can also be symbolized algebraically and it can be abstracted into the world of axioms and theorems. We problematize this multiple nature of symmetry which on one hand is supportive and on the other acts as a contributory factor to learners' gaps in knowledge. Tall's three worlds of mathematics helped us to show the transition of symmetry from the embodied through symbolic to the formal world and the inherent gaps attributed to the shifts in thinking thereof. We then used this same framework to analyse learners' responses to a reflective symmetry task. The results show that many learner responses could be explained explicitly by the lack of flexibility in the applicability of experiences in the embodied world of reflective symmetry. Learners' responses were deeply rooted in the embodied world, which indeed remains helpful in some situations but tended to confuse them in others, hence inhibiting further application. The article recommends that teachers need to understand these subtle changes so that they can address the challenges explicitly.

Keywords: learning paradox; embodied; symbolic; formal; reflective symmetry

Introduction

In mathematics and many of its applications, great importance is placed on the idea of symmetry, hence many curricula globally set symmetry as one of the significant geometry concepts that students need to grasp early on. Before learning it at school, children experience reflective symmetry from a very early age because it is an aspect of their bodies, of nature and of many man-made constructions. In this sense, Tall (2008) would argue that the study of reflective symmetry at school is rooted within operations on the environment or manipulation of physical objects. Once these operations on physical objects are routinized, they are then symbolized and as the focus shifts from the manipulation of physical objects to manipulation of symbols, mathematical thinking also shifts from the embodied to the symbolic world. The later transition to the formal axiomatic world builds on these experiences of embodiment and symbolism to formulate formal definitions and to prove theorems using mathematical proof. Hence, when students begin to study formal proof of symmetry or any other concept at school or university, Tall (2004) would argue that they already have a wealth of prior knowledge (met-befores) that builds from the embodiment of physical conceptions and actions into abstract mathematical concepts.

Since the concept of symmetry is heavily embedded in the everyday, there is a strong belief that its study might help students to draw upon their prior everyday knowledge in order not only to make sense of the more refined scientific form of symmetry but also to connect or apply this more refined form elsewhere. Despite this assumed symbiotic relationship, research shows that when reflective symmetry is taken from its everyday context into its more scientific/mathematical form, a large portion of high school learners continue to exhibit very limited understanding (Bulf, 2010; Hoyles & Healy, 1997; Xistouri, 2007). Given such well-documented evidence of discontinuities when learners use such prior knowledge in school, we then mused: 'Is there a potential gap between the informal/everyday and formal knowledge of reflective symmetry? If so, what is the nature of the gap and in what ways is informal knowledge likely to be simultaneously necessary and problematic for students?'

Locating our Strategic Position

Gray, Pitta, and Tall (2000) placed such discontinuities within the context of a 'proceptual divide', which they argued revealed a catastrophic difference between those children who processed information in a flexible way and those who invoked the use of physical or visual objects. The problem of the 'proceptual divide' (Gray et al., 2000), 'learning paradox' (Bereiter, 1985) or the 'Great Divide' (Roschelle, 1997) has attracted the interest of many researchers who in turn have approached its study differently. In an attempt to offer a solution to the problem, Gray et al. (2000) cautioned that we should not succumb to irresolvable contradictions created by the never-ending debates about prior knowledge that have dominated these divides. On one side research tends to characterize prior knowledge as an epistemological obstacle (Bachelard, 1938), while on the other side constructivism appears to take for granted that new levels of mathematical thinking are necessarily built logically and consistently on previous experience (Hoyles & Healy, 1997). It is McGowen and Tall's (2010) contention that neither side on its own has resolved the problem of the learning paradox because prior knowledge appears to be simultaneously necessary and problematic. They suggest that this didactic problem is not resolved by hiding one of the parts, but that both the necessary and problematic aspects need to be objectified because the links between the informal and formal knowledge in mathematics are inevitable. Similarly, Bergsten (1999) argued that these problems could not be worked out without profiting from taking the other into account—the intuitive and formal always go together, whether separated or linked. Gray et al. (2000) therefore suggested that we need to determine when this prior knowledge is necessary and when it is problematic, so that we may provide necessary support both to those who develop flexibly and also to those who, at the very start of their mathematical development, appear to traverse a cognitively slow route. We align our work with Gray et al.'s view that prior knowledge of reflective symmetry is simultaneously necessary and problematic.

In our previous contribution with a similar focus (Mhlolo & Schäfer, 2013), we showed how textbooks 'as teachers at a distance' (Fauvel, 1991) might have contributed to the way learners poorly conceptualized reflective symmetry. For this article, we use different lenses and shift the focus from books, teachers and learners to the nature of reflective symmetry itself. This follows McGowen and Tall's (2010) concern that much of the research on why students construct idiosyncratic and non-conforming understandings in mathematics places emphasis on 'external' factors that include, inter alia, textbooks, negative images of mathematics teachers, social factors, parents and others. In all such studies there are rarely any references to problematic factors that may arise from the 'internal', i.e. the nature of the mathematics content itself. Gray et al. (2000) had earlier conjectured that the nature of the object that was dominant in the child's imagery and how that imagery is used within elementary mathematics may be a determinant associated with achievement. Extensive research has documented critical moments where an earlier and helpful way of thinking about a concept fails to account for new ideas, which in turn has far-reaching consequences on the development of children's reasoning (Bruner, Oliver, & Greenfield, 1966). Consistent with such observations, McGowen and Tall (2010) argued that explanations of students' unconventional interpretations of mathematical concepts based on external factors (books, teachers, learners, parents, social backgrounds) would have been built on a deficit model and would be insufficient if such explanations did not consider prior knowledge arising from the way imagery is used within elementary mathematics. Gray et al. (2000) place their work within theories accounting for the transformation of processes into concepts that have helped them to shift attention from doing mathematics to conceptualizing mathematics.

Objectives of this Article

This article is divided into two main parts. The first part aims to establish the dominant imagery of reflective symmetry that gets foregrounded in the embodied/informal mathematical world. The second part, which is empirical, looks at evidence in learners' written responses to a task on reflective symmetry that could support the theory that such dominant imagery would be simultaneously supportive and problematic.

Conceptualizing the Transition of Reflective Symmetry from the Informal to Formal

Substantial interest in the cognitive development of mathematics has focused on the relationship between actions on physical properties of particular objects and how these actions or operations become thematized objects of thought or entities. Many frameworks are available to present this long-term journey from physical perception and action to more complex ideas, but there is evidence to suggest that Tall's (1969–2014) work on this subject has appealed to many other mathematics researchers. After considering many other frameworks, Gray and Tall (2001, p. 70) wrote:

> For several years […] we have been homing in on three […] distinct types of concept in mathematics. One is the embodied object, as in geometry and graphs that begin with physical foundations and steadily develop more abstract mental pictures through the subtle hierarchical use of language. Another is the symbolic procept which acts seamlessly to switch from a mental concept to manipulate to an often unconscious process to carry out using an appropriate cognitive algorithm. The third is an axiomatic object in advanced mathematical thinking where verbal/symbolical axioms are used as a basis for a logically constructed theory.

These threads have become known as Tall's (2008) 'three worlds of mathematics' described briefly as follows:

1. An object-based *embodied* world—relating to our sensory perceptions of and the physical actions on the real world.
2. An action-based *symbolic* world—relating to our use of symbolism in arithmetic, algebra and more general analytic forms that enable us to calculate and manipulate to get answers.
3. A property-based *formal* world—relating to the formal axiomatic world of mathematicians which is the final bastion of presentation of coherent theories and logical proof. (Tall, 2008, p. 7)

Tall (2008) captured this development from the embodied through symbolic to the formal world succinctly in his model shown in Figure 1.

The three worlds describe a hierarchy of qualitatively different ways of thinking that individuals develop as new conceptions are compressed into more thinkable concepts. The embodied world, containing embodied objects (Gray & Tall, 2001), is where we think about the things around us in the physical world, and it 'includes not only our mental perceptions of real world objects, but also our internal conceptions that involve visuo-spatial imagery' (Tall, 2004, p. 30). The symbolic world is the world of procepts, where actions, processes and their corresponding objects are realized

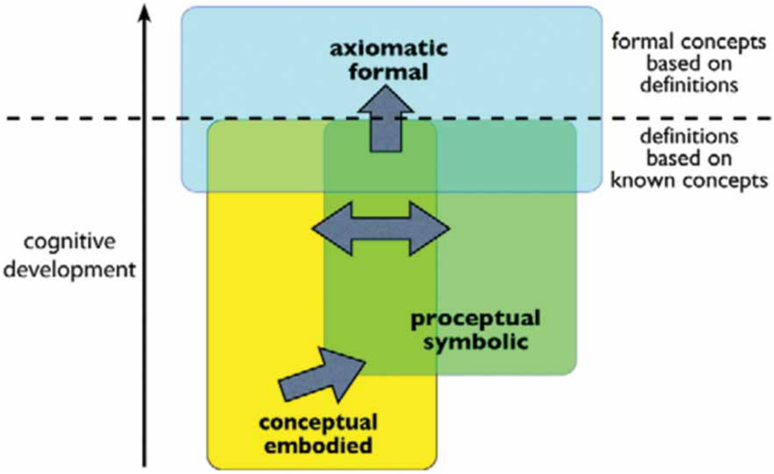


Figure 1: Development through three worlds of mathematics (Tall, 2008)

and symbolized. The formal world of thinking comprises defined objects presented in terms of their properties, with new properties deduced from objects by formal proof. We now show how this framework resembles the transition of reflective symmetry from its informal everyday world to the formal mathematical one.

The Embodied World of Reflective Symmetry

There appears to be general consensus that our mathematical growth of reflective symmetry develops through a lifetime of experiences that we gain from both powerful everyday examples as well as through formal mathematics instruction (Tall, 2004). Several studies suggest that humans have an innate cognitive mechanism related to symmetry reasoning and observations are that symmetric stimuli are not only preferred but are consistently detected faster, discriminated more accurately and often remembered better than asymmetrical ones (Clements, 2004). Children also experience symmetry from an early age because it is an aspect of nature. Beyond the innate and early experience of symmetry in nature, the topic is formally introduced in school mathematics. Generally reflective symmetry is introduced by folding and matching, connecting the matching points and measuring the distance from each point to the fold line. Similarly we were also interested to see how reflective symmetry was introduced in the South African recommended mathematics textbooks that were used by learners in our research schools.

Our observations (Mhlolo & Schäfer, 2013) were that reflective symmetry is introduced informally through looking at patterns and identifying ‘geometrical shapes’, which are then reflected (folded), mainly around a vertical and horizontal axis. These textbooks state that when a figure has an axis of symmetry, it will reflect onto itself if it is reflected about such a line. The activities that follow this definition involve copying diagrams and drawing a line of reflection for each one. From the way both the definition and properties of reflective symmetry are presented, one can see the dominance of an object-based perspective where reflective symmetry is viewed in terms of a ‘shape/diagram/figure’ that is of the ‘same size’ or ‘congruent’, which is ‘reversed’ or ‘flipped over’ or ‘laterally inverted’ in the mirror line.

The Symbolic World of Reflective Symmetry

Once perceptions and operations on physical objects are routinized, they are then symbolized so that they can be performed without much conscious effort or link with the real world. Taking transformation matrices as one way of symbolizing reflective symmetry, let us use the following three points A (-3; 6), B (4; -2) and C (-5; 1) that were in our task to demonstrate the different effects of the symbolized process as shown in Table 1.

Table 1: Matrices and their different reflections

Transformation matrix	Effect of matrix	Symbolic evidence
$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	This transformation matrix creates a reflection in the x-axis where the x-coordinate remains unchanged, while the y-coordinate changes sign	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$
$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	This transformation matrix creates a reflection in the y-axis where the y-coordinate remains unchanged, while the x-coordinate changes sign	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	This transformation matrix creates a reflection in the line y = x where the x-coordinate becomes the y coordinate and y-coordinate becomes the x-coordinate	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$

The results of the operations can also be generalized and symbolized algebraically, as shown in Table 2.

Table 2: Algebraic formulas for reflective symmetry.

Reflection	Rule
About the x-axis	(x; y) becomes (x; -y)
About the y-axis	(x; y) becomes (-x; y)
About y = x	(x; y) becomes (y; x)

Mathematics makes use of such symbolic notations, which serve a dual role as an instrument of communication and of thought. Hence Tall (2008) used the term 'procepts' to refer to such symbolic notations, which become thinkable objects that function both as processes to do reflections and as concepts to think about.

The Formal World of Reflective Symmetry

Later on the concept of reflective symmetry is abstracted so that it is not seen as a static property of shapes but as a function with specific properties that transcend any particular configuration of object and mirror line. The formal world is a world of axioms, definitions and theorems. Statements are true because they can be proved from the axioms and definitions by formal deduction. For example, the introduction of coordinate geometry by René Descartes (1596–1650) opened up great links between geometry and algebra. In this system a point's location on the Cartesian plane is given by two numbers (x; y), a technique that enables us to find distances between points, mid-points, slopes, equations, perpendicularity, transformations, areas and perimeters. For instance, if we wanted to reflect the point A (-3;6) in the y-axis without drawing any visual aid like a Cartesian plane, we could start by arguing that we are looking for a point A′ that lies on a line perpendicular to the y-axis (mirror line) and the same distance from A but on the opposite side. The equation of the line we are looking for, which must be parallel to the x-axis with a gradient of 0, is therefore y = 6. On this line, the line segment $\overrightarrow{AA'}$ that we are particularly interested in has the point (0; 6) as its mid-point. Point A′ (-3; 6) has an absolute distance of 3 units from this mid-point (0; 6), implying that A′ is also 3 units but on the other side of the y-axis (the mirror line). Using the distance formula we can find this image position thus:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Where:

$$6 = \sqrt{(x_2 - -3)^2 + (6-6)^2} \Rightarrow 36 = (x_2 + 3)^2$$

$\therefore x_2 = 3$, implying that A′ (3; 6).

So we can see how through specifying axioms and definitions of parallel lines, gradient, perpendicularity and mid-points of linear equations we are able to locate the images of such points under reflections without resorting to sensory perceptions and physical actions on the objects. We have many such cases where learners are required to work with specific properties of reflective symmetry that transcend any particular configuration of object and mirror line.

Possible Gaps in Conceptual Change of Reflective Symmetry

As one might notice, this transition of reflective symmetry from the embodied through symbolic to the formal worlds then leads to very different methods of making arguments, or what Rodd (2000, p. 225) described as 'warrant for truth for that which secures knowledge'. In the embodied world, it occurs first through 'seeing' that something is true—*we can see it.* In the symbolic or proceptual world, it arises through calculating a correct result, or using the generalized arithmetic of algebraic manipulation to verify the required symbolic statement—*we can calculate it*. In the formal world it arises through specifying axioms and definitions set-theoretically—*it is an axiom*.

A major gap has been attributed to the shifts in thinking and warrants for truth as the concept transforms from the embodied through the symbolic to the formal world of reflective symmetry. In the embodied world of reflections the learner is likely to be easily convinced about its validity visually, since the results of both seeing real world examples and folding are produced by a practicable and observable process. On the other hand, the arithmetic and algebraic computations deriving from the symbolic and formal worlds may be insufficient because the warrant for truth is not visible in that the evidence does not reveal itself in a manner as obvious as the diagrammatic/visual solution. A common problem—and probably the most severe gap in students' conceptualization of mathematics—is associated with this inability to effectively abstract maths concepts from the concrete or real world. Many weak students prefer to argue on the basis of empirical evidence, though many are not able to use the evidence in a mathematically appropriate way. For example, Gray et al. (2000) observed that low achievers had a tendency to concretize and for these children mathematics involved action where they used 'real things', while high achievers appeared to focus on abstractions. Tall (2006) captures the gap caused by this change from the embodied to the formal metaphorically as he says that the formal-axiomatic world expresses properties in general set-theoretic terms to 'turn mathematics on its head'. In this study we were particularly watchful for this gap between low and high achievers caused by the former's persistent reliance on the object-based approach.

Methodology

Sampling Process

This article draws from our five-year research project which aimed to investigate high school learners' problems in mathematics. The 235 Grade 11 learners (± 17-year-olds) from the 13 schools that we work with in this article are located in previously disadvantaged communities within a 60 km radius of Grahamstown in the Eastern Cape Province of South Africa. A benchmark test was given covering many topics on the curriculum, but this article focuses on responses to a task on reflective symmetry. For this article we analysed all 235 learner responses.

Ethical Clearance

Because of the large number of learners (± 1000) involved in this project and the cumbersome legal requirements involved in getting consent from minors or parents, we were unable to get ethical clearance to interview learners. Our observations were therefore based solely on test results. While this might be viewed as a limitation of the study, we have examples of many other large-scale studies that have influenced policy based on written responses by learners. This suggests that there is a lot we can learn from analysing learners' written responses.

Research Instrument

We collected data from learners through a task on reflective symmetry whose questions were phrased as follows:

The diagram below shows a Cartesian plane with points A (-3; 6), B (4;-2) and C (-5;1). Draw on the grid below, A′, B′, and C′, clearly labelling the coordinates of each point if:

(a) A′ is the image of A reflected in the y-axis.
(b) B′ is the image of B reflected in the line y = 0.
(c) C′ is the image of C reflected in the line y = x.

When we examined the characteristics of this task, our view was that it offered us an opportunity to see how learners borrowed from the everyday, i.e. a visual approach that advocates investigations and discovery of properties via concrete manipulations, models and diagrams. It was also possible for learners to tackle the task using both a symbolic and formal approach by applying the general rules for reflective symmetry. We also conjectured that learners might also have complemented the three approaches (visual, symbolic and formal) in order to verify the accuracy or correctness of the image positions after reflecting in different axes.

Analytical Tool

Stewart and Thomas (2009) are among the many researchers who have used Tall's framework for analysing learner responses. In this article we borrow from a framework that they constructed to examine a learner's thought processes on the concept vector and scalar multiplication in each of the three mathematical worlds of embodied, symbolic and formal. We modified the indicators under algebraic representation in order to capture the specific algebraic symbolization for reflective symmetry as shown in Table 3.

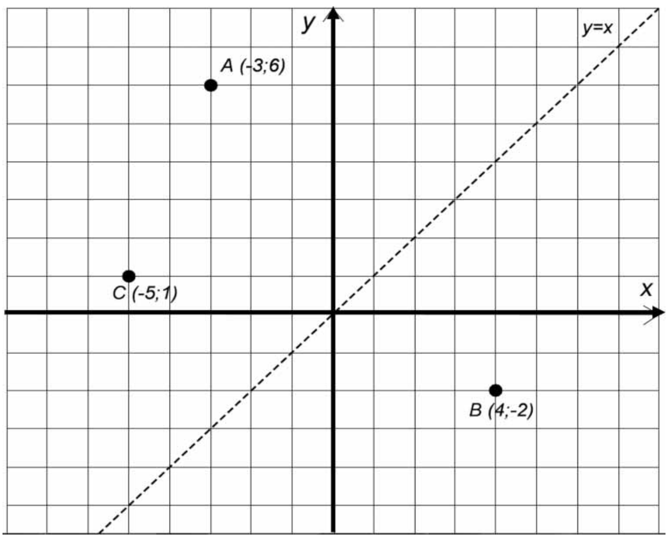
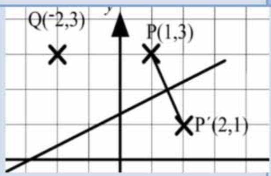


Figure 2: Task on reflective symmetry

Table 3: Reflective symmetry as seen through Tall’s three worlds (adapted from Stewart & Thomas, 2009).

Embodied world	Symbolic world		Formal world
	Algebraic representation	*Matrix representation*	
 The embodied world is where learners think about things around them in the physical world. In this task we identified it when learners construct lines to show the mirror line or to join points or form shapes, diagrams or figures.	(x; y) becomes (x; -y) (x; y) becomes (-x; y) (x; y) becomes (y; x) The use of such algebraic notation for solving the task would be indicative of a learner working in the symbolic world.	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ The use of such matrices for solving the task would be indicative of a learner working in the symbolic world.	For this task, finding gradient, distances between points, midpoints, equations, perpendicularity, through formal proofs and theorems e.g. use of the Pythagoras theorem in the distance formula, would be indicative of a learner working in this formal world.

This tool then enabled us to categorize the responses into the three worlds as discussed.

Results and Discussion

We took sample pictures to enable us to make a more comprehensive discussion. Consistent with our objective, we were interested to identify learners’ representations that could be associated with these three worlds of mathematics.

Responses Classified under Object-Based Embodied World

Our immediate observation was that the entry point for 199 (85%) of these 235 learners was through joining the three points A, B and C to form a triangle as shown in Figure 3. That such regularity was observed across 13 different high schools with 13 different teachers, with a consistency far beyond chance, echoes the concern of Bereiter (1985, p. 224):

> Out of a magnitude of correspondences that might be noticed between one event and another, how does it happen that … .different children, with consistency far beyond chance, tend to notice the same correspondences?

Why were learners noticing a triangle when there was no triangle asked for? McGowen and Tall (2010) would argue that this is not simply a question of the learner ‘making a mistake’ because these meanings are naturally generated in our social, intellectual and physical environment. With specific reference to reflective symmetry, Hoyles and Healy (1997) made similar observations and posited that the very existence of these regularities suggested that learners had constructed a socially shared understanding of reflection. So by joining points to form a ‘shape’, in our view they were using non-mathematical clues they had constructed from both their everyday experiences and their previous

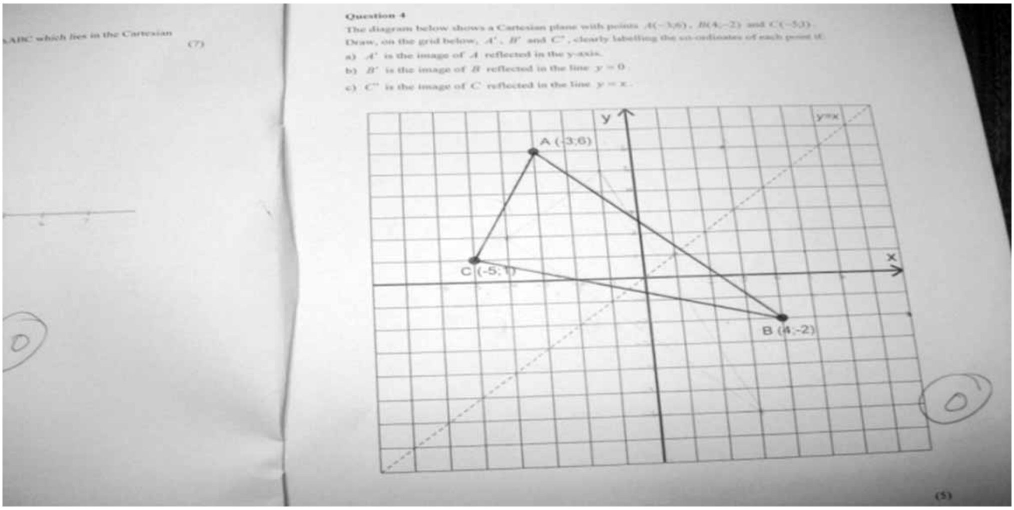


Figure 3: Learners connecting all three points to form a triangle

work in school. Judging by the different ways through which the concept is first encountered, the learners were using a well-formed conception that reflection suggests a shape of the 'same size' or 'congruent' to the image. So the learners had a conception of reflective symmetry which they had 'met-before' and which continues to be supportive in some cases but problematic in other contexts, such as this particular one.

The second regularity that we noticed was that 101 learners (43%) then went on to reflect their triangles mainly in the y-axis and in the x-axis, as shown in Figures 4 and 5 respectively.

Consistent with Gray et al. (2000), we see in these responses a dominant focus on objects and actions with and on those objects. We attribute this to the rich set of meanings around symmetry that is developed outside school, which according to Xistouri (2007) shapes student responses in the mathematics classroom. For example, in the early activities on reflective symmetry, paper folding is usually utilised to assist learners in the doing of a reflection. The responses we see here are typical of paper folding. In the less resourced schools, paper folding might be a supportive met-before in that they equip learners with mental tools, which can be used in new problem situations. Research has shown, however, that such tools might also have side effects (problematic met-befores) in terms of transferability to new situations such as this one. For example, what visual evidence would convince the learner that an image point is of the same shape and size as the original point or that the image point has been flipped over? Here is a typical example where an idea built in the embodied world (met-before) fails in another and may need significant rethinking in this new context.

Despite their failure to identify the object that was to be reflected, their strategies follow consistently from a definition of reflection in the embodied world. According to Hoyles and Healy (1997), if we focus on the regularities among the responses rather than on the errors, it is clear that the learners exhibit well-reasoned constructions. We can see a common set of invariant properties: the reflected images are the same size and shape as the original triangle that they drew; they are the same distance away from the mirror as the image, and they are 'opposite' or 'reversed'. Roschelle (1997) posits that many 'misconceptions' are correct elements of knowledge that has been over-generalized and so by specifying a narrower range of situations the concept becomes correct. In fact, we argue that if this task had required learners to reflect a given triangle (narrowing the range of situations) instead of three independent points in three different mirror lines, these 101 learners would have got a correct solution. This also confirms McGowen and Tall's (2010) argument that they had preconceptions and not misconceptions.

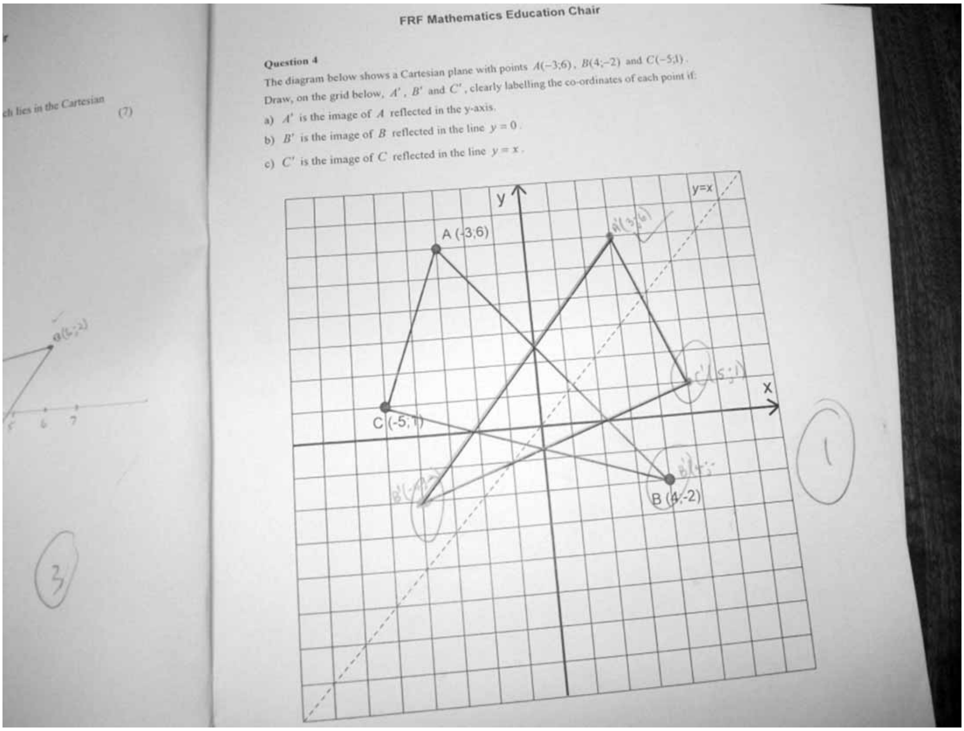


Figure 4: Learners reflecting in the y-axis

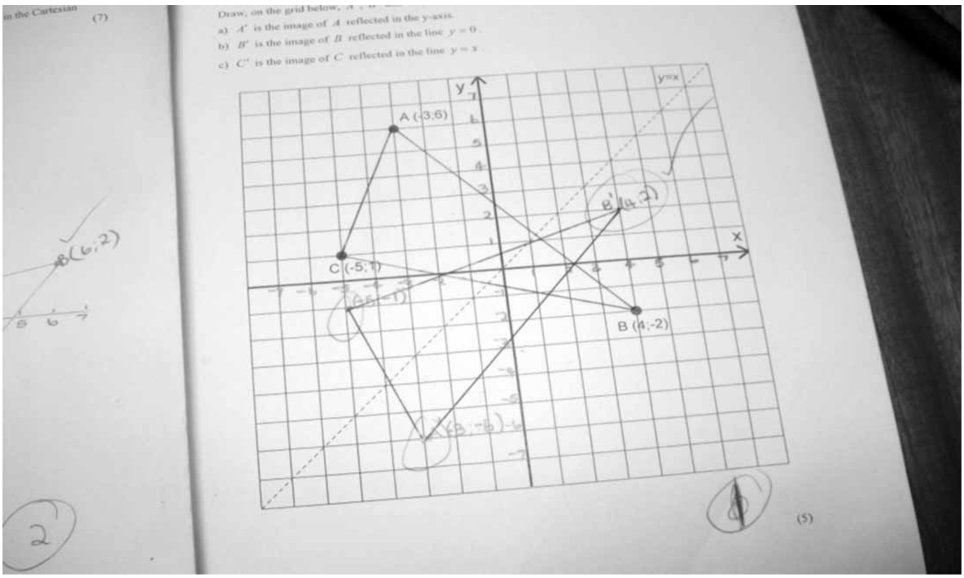


Figure 5: Learners reflecting in the x-axis

As to why learners could not validate their solutions using algebraic formulas, Fischbein (1987) would argue that the constructions had intrinsic certainty intuitively. He defined intuition as a kind of cognition that is accepted directly without feeling that any kind of justification is required. Intuition is often linked to common sense and is characterized by intrinsic certainty, self-evidence and immediacy as opposed to scientific knowledge. This further supports our argument that the learners' met-befores were located in the embodied world. In the embodied world, a warrant for truth occurs first through 'seeing' that something is true, and just by looking at their constructions one does indeed see a 'correct' reflection. There is evident 'truth' suggested by their met-befores, hence they were convinced their solutions were correct.

Learners Avoiding the Line y = x

A regularity that we also noticed was that of the 85% who drew triangles and the 43% who then went on to reflect in the x-axis and y-axis, none of them tried to reflect in the line y = x. We felt that this observation deserved special mention as other studies have made similar observations (Hollebrands, 2004; Hoyles & Healy, 1997). This regularity can also be traced back into the embodied world. The word symmetry is used in everyday language to denote reflection. When one says that something is symmetrical it is implicitly understood that reflection symmetry is meant. In the real world mirror reflection generally appears in two different ways: horizontal reflection and vertical reflection. Animals, humans, insects, plants and buildings are usually used as the simplest and most common examples of all symmetries; they all show vertical bilateral symmetry. From the real world an object reflected in water is usually exemplified as having horizontal line symmetry along the waterline. When symmetry is introduced in school it is also common that the first activities that the pupils are exposed to are about vertical and horizontal reflective symmetry. So the real world gives a familiar image and a common meaning of symmetry that is depicted in horizontal and vertical bilateral symmetry. Hence in our view both the common meaning and the social representation of reflective symmetry are supportive met-befores in that they provide tools for the learning of symmetry but do not provide the learner with an image of an inclined bilateral symmetry. Humans follow an unconscious bias towards linking concepts to the nodes they already know and such links provide a common basis for understanding the concepts attached to the nodes.

Bulf (2010) points to Bachelard's (1938) concept of 'epistemological break', which underlines the discontinuity at work in the history of sciences (including mathematics). Bulf's observation was that there is an inherent contradiction between the formal and the embodied worlds. Within this contradiction he identified the obstacle of 'the excessive use of familiar images' or the obstacle of 'common meaning' and 'social representations'. Similarly, Kimmel (2002) observed that if a source domain is used to shed light on one or more salient targets, this increases its likelihood to be chosen as a source domain in the future; it becomes a cultural attraction of meaning.

Responses Classified under Symbolic Embodiment

We observed that 10 learners used the algebraic formulas for reflective symmetry as described earlier but no learner used the matrix method. Of the 10 learners who used algebraic formulas, only 6 used the method correctly. The remaining 4 made some errors but earned some part-marks.

Our observations are summarized in Table 4.

A total of 26 learners did not attempt this task at all, therefore we could not classify them. A total of 199 were classified under the object-based embodiment world. These comprised 98 learners who joined the three points to form a triangle but could not proceed (no marks awarded) and 101 learners who joined the three points to form a triangle then went on to reflect in the y-axis and the x-axis (part-marks awarded). Only 6 learners were able to provide a correct solution to the task through the use of the algebraic symbolic rules. No learner attempted to use any of the formal theorems of reflective symmetry to solve the task.

Table 4: Learner responses classified under Tall's three worlds of reflective symmetry (n = 235)

Unclassified			Embodied world			Symbolic world			Formal world		
26			199			10			0		
Correct	*Partially incorrect*	*Incorrect*	*Correct*	*Partially correct*	*Incorrect*	*Correct*	*Partially incorrect*	*Incorrect*	*Correct*	*Partially correct*	*Incorrect*
0	0	26	0	101	98	6	4	0	0	0	0

Implications

We see our work as contributing to theory, curriculum design and practice in a number of ways. In terms of theory, this study is located in the broader constructivist discourse. Constructivism's popularity is largely due to consensus that the learner is not a passive recipient of knowledge but that knowledge is 'constructed' by the learner in some way in the process of interacting with a concept (Rowlands & Carson, 2001). While this maybe uncontroversial, controversy begins when the interpretation of constructivism is taken to suggest that this construction of knowledge by the learner is 'unproblematic'. From a constructivist perspective our analyses show that

> … concepts are not inherent in things but have to be individually built up by reflective abstraction; and reflective abstraction is not a matter of looking closely but of operating mentally in a way that happens to be compatible with the perceptual material at hand. Hence physical materials are indeed useful, but must be seen as opportunities to reflect and abstract, not as evident manifestations of desired concepts. (Von Glasersfeld, 1995, p. 184)

Our analyses therefore point to the need to reinterpret basic assumptions in the constructivist discourse. Empirical evidence abounds that shows that in the apparent logical structure of mathematics curricula, the biological brain will bring previous experiences to interpret the situations that are presented, which can lead to unforeseen difficulties that arise through apparently sensible approaches.

In building a curriculum, designers usually focus on the positive effect of met-befores, such as the way everyday examples of reflective symmetry generalize to the formal understanding. However, there is far less emphasis on strategies for dealing with the negative effects of met-befores that no longer work in a new context. Yet learners' responses in this study showed how the introduction of inevitable met-befores of reflective symmetry at an informal/everyday stage will result naturally in the need to address what is necessary when shifting to the formal world. Thus, reorganization of knowledge of symmetry is an important part of curriculum building. At present it is an idea that is almost totally absent from most curriculum frameworks (Tall, 2008).

In terms of practice, many countries have for many years encouraged all mathematics teaching to be based on practical activities. However, the results of this study show, consistent with many other instances, how practical experiences help to build up a coherent overall picture, but may contain implicit elements that act as impediments in future learning. Educators therefore have a responsibility to view more than the positive met-befores that are seen to be pre-requisites for learning new mathematics. It is essential that they also consider problematic met-befores that impede learning and enter into a dialogue with students to encourage them to develop new ways of working in new contexts. When teachers do not understand the significance of these subtle changes, their students can become very confused as their knowledge structures become increasingly fragmented and lacking in coherence. The solution may not be simply for the teacher to be more enthusiastic and work towards positive attitudes but to address the problematic met-befores.

Concluding Remarks

Consistent with our objectives, we showed how the development of reflective symmetry progresses from the embodied through the symbolic to the formal world. Gaps in knowledge often arise as the thinking shifts from the embodied through to the formal worlds and indeed responses analysed in this article suggest that learners were reasoning mainly from their experiences in the embodied world; experiences which then gave rise to 'met-befores' that inhibited thinking in a different context. The recommendation we make is consistent with the suggestion made by McGowen and Tall (2010) that the formal world of reflective symmetry requires a considerable change in approach in which the learner must build on the met-befores of embodiment and symbolism in elementary mathematics which need rethinking to give the formal proof of axiomatic theories.

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